Functional Analysis (WISB315)

Exercise sheet 9

Exercises marked with a ✍ symbol are **hand-in exercises**. The exercises marked with a + are particularly important for understanding the material. Exercises marked with a ⏱ are quick warm-up exercises.

+ **Exercise 1** (Lemma 6.11)**.** Let $\mathcal{H}$ and $\mathcal{K}$ be a Hilbert space and let $T \in B(\mathcal{H}, \mathcal{K})$. Show that:

(i) $\operatorname{Ker} T = (\operatorname{Im} T^*)^\perp$:

(ii) $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$;

(iii) $\operatorname{Ker} T^* = \{0\}$ if and only if $\operatorname{Im} T$ is dense in $\mathcal{K}$.

⏱ **Exercise 2** (Lemma 6.14)**.** Let $\mathcal{H}$ be a Hilbert space. Show that if $T \in B(\mathcal{H})$ is invertible, then T^* is invertible with inverse $(T^*)^{-1} = (T^{-1})^*$.

Exercise 3 (Exercise 6.5 in the book)**.** Let $\mathcal{H}$ be a Hilbert space and let $T \in B(\mathcal{H})$.

(i) Show that $\operatorname{Ker} T = \operatorname{Ker}(T^*T)$.

(ii) Deduce that $\overline{\operatorname{Im} T^*} = \overline{\operatorname{Im} T^*T}$

In this exercise sheet, all vector spaces are over $\mathbb{C}$.

Exercise 4. Let $\mathcal{H} = \mathbb{C}^2$, equipped with the standard inner product

$$\langle x, y\rangle = x_1\overline{y}_1 + x_2\overline{y}_2, \quad x, y \in \mathcal{H}.$$

(i) Find $T \in B(\mathcal{H})$ that is self-adjoint, but not unitary.

(ii) Find an operator $T \in B(\mathcal{H})$ that is unitary, but not self-adjoint.

(iii) Find an operator $T \in B(\mathcal{H})$ that is normal, but neither self-adjoint nor unitary.

(iv) Find an operator $T \in B(\mathcal{H})$ that is not normal.

Exercise 5 (Example 6.17)**.** Show that the unilateral shift $S \in B(\ell^2)$ (see (4.2) in the book) is not normal.

Exercise 6 (Example 6.38)**.** Let $S \in B(\ell^2)$ be the unilateral shift. Show that:

(i) If $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ then λ is an eigenvalue of S^*.

(ii) $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

Exercise 7 (Example 6.17)**.** Show that the unilateral shift $S \in B(\ell^2)$ (see (4.2) in the book) is not normal.

Exercise 8 (Exercise 6.23 in the book)**.** Let $T : \ell^2 \to \ell^2$ be the operator defined by

$$T(x_1, x_2, x_3, x_4, \dots) = (0, x_1, 0, x_3, 0, \dots), \quad (x_n) \in \ell^2.$$

Show that $T \in B(\ell^2)$ and $\sigma(T) = \{0\}$.

We recall that the resolvent set of $T \in B(X)$ is $\rho(T) := \mathbb{C} \setminus \sigma(T)$.

+ **Exercise 9** (the resolvent set is open)**.** Let X be a Banach space, $T \in B(X)$, and $\lambda_0 \in \rho(T)$. Suppose $\lambda \in \mathbb{C}$ is such that

$$|\lambda - \lambda_0| < \left\|(\lambda_0 - T)^{-1}\right\|^{-1}.$$

Show that $\lambda \in \rho(T)$ and

$$(\lambda - T)^{-1} = \sum_{i=0}^{\infty} (\lambda_0 - \lambda)^i \left((\lambda_0 - T)^{-1}\right)^{i+1},$$

where the series converges in the operator norm sense.

+ **Exercise 10** (Theorem 6.39 (a))**.** Let $\mathcal{H}$ be a Hilbert space and let $T \in B(\mathcal{H})$. Show that if p is a polynomial then

$$\sigma(p(T)) = \{p(\mu) : \mu \in \sigma(T)\}$$

Remark. *We can also use the shorthand notation* $p(\sigma(T)) := \{p(\mu) : \mu \in \sigma(T)\}$*.*

⏱ **Exercise 11** (Theorem 7.3)**.** Show the following two statements.

(i) Let X, Y be normed spaces, $S, T : X \to Y$ compact operators, and $a, b \in \mathbb{C}$. Then $aS + bT$ is a compact operator.
(ii) X, Y, Z normed spaces, $S \in B(X, Y)$, and $T \in B(Y, Z)$ such that S or T is compact. Then $TS : X \to Z$ is compact.

+ **Exercise 12** (Theorem 7.5)**.** Let X, Y be normed spaces. Show the following:

(i) If $\dim Y < \infty$ then every $T \in B(X, Y)$ is compact.
(ii) If $\dim X < \infty$ then every linear $T : X \to Y$ is compact.

✍ **Exercise 13** (exercise similar to Example 7.11)**.** Show that the operator $T : \ell^2 \to \ell^2$ defined by

$$(Tx)_n = n^{-3} x_n, \quad \forall x = (x_n)_{n \in \mathbb{N}} \in \ell^2,$$

is well-defined, bounded and compact.

Exercise 14 (Exercise 7.7 in the book)**.** Let $\mathcal{H}$ be an infinite-dimensional Hilbert space with an orthonormal basis (e_n), and let $T \in B(\mathcal{H})$. Show that if T is compact then $\lim_{n \to \infty} \|Te_n\| = 0$.
