Exercise 2. For each $N \in \mathbb{N}$ we define the functional $M_N : \ell^\infty(\mathbb{Z}) \to \mathbb{C}$ by

$$M_N(x) = \frac{1}{2N+1} \sum_{|n| \leq N} x_n, \quad \forall x \in \ell^\infty(\mathbb{Z})$$

For each $a \in \mathbb{Z}$ we define the operator $T_a : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ by

$$(T_a x)_n = x_{n+a}, \quad \forall x \in \ell^\infty(\mathbb{Z}).$$

Let also $e \in \ell^\infty(\mathbb{Z})$ be defined by $e_n = 1$ for all $n \in \mathbb{Z}$, and let $V \subset \ell^\infty(\mathbb{Z})$ be the subspace of all x that can be written as a finite sum:

$$x = \lambda_0 e + \sum_{j=1}^{p} \lambda_j \left(f_j - T_{a_j} f_j\right)$$

for some (x-dependent) $p \in \mathbb{N}, \lambda_0, \ldots, \lambda_p \in \mathbb{C}, a_1, \ldots, a_p \in \mathbb{Z}$, and $f_1, \ldots, f_p \in \ell^\infty(\mathbb{Z})$.

(i) [1 p.] Show that for all $x \in \ell^\infty(\mathbb{Z})$ and $a \in \mathbb{Z}$,

$$|M_N(x - T_a x)| \leq \frac{2|a|}{2N+1} \|x\|_\infty.$$

(ii) [0,5 p.] For each $x \in V$, show that $\lim_{N\to\infty} M_N(x)$ equals the coefficient λ_0 in any decomposition (1). In consequence, we can define a map $L : V \to \mathbb{C}$ by

$$L(x) = \lambda_0.$$

Show that $L \in V'$. (iii) [0, 5 p.] Show that the functional L has norm $\|L\| = L(e) = 1$.

(iv) [1, 5 p.] Show that there exists a functional $M \in (\ell^\infty(\mathbb{Z}))'$ such that:

(a) $M(x) = L(x)$ for all $x \in V$,

(b) $M(e) = \|M\| = 1$,

(c) $M(x) = M(T_a x)$ for all $a \in \mathbb{Z}$ and $x \in \ell^\infty(\mathbb{Z})$.

(v) [0, 5 p.] Show that there exists no $y \in \ell^1(\mathbb{Z})$ such that

$$M(x) = \sum_{n \in \mathbb{N}} x_n y_n, \quad \forall (x_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$$

(vi) [0, 5 p.] Using the result from the previous question show that $\ell^1(\mathbb{Z})$ is not reflexive.

In the next exercises, all vector spaces are over $\mathbb{C}$.

(i) Observe that if $a \geq 0$

$$|M_N(x - T_a x)| = \frac{1}{2N+1}\left|\sum_{|n|\leq N}(x - T_a x)_n\right| = \frac{1}{2N+1}\left|\sum_{n=-N}^{N} x_n - \sum_{n=-N+a}^{N+a} x_n\right| \tag{1}$$
$$= \frac{1}{2N+1}\left|\sum_{n=-N}^{-N+a-1} x_n + \sum_{n=-N+a}^{N} x_n - \sum_{n=-N+a}^{N} x_n - \sum_{n=N+1}^{N+a} x_n\right| \tag{2}$$
$$= \frac{1}{2N+1}\left|\sum_{n=-N}^{-N+a-1} x_n - \sum_{n=N+1}^{N+a} x_n\right| \leq \frac{1}{2N+1}\left(\sum_{n=-N}^{-N+a-1}|x_n| + \sum_{n=N+1}^{N+a}|x_n|\right) \tag{3}$$
$$\leq \frac{1}{2N+1}\left(\sum_{n=-N}^{-N+a-1}\|x\|_\infty + \sum_{n=N+1}^{N+a}\|x\|_\infty\right) = \frac{1}{2N+1}\left(a\|x\|_\infty + a\|x\|_\infty\right) \tag{4}$$
$$= \frac{2a\|x\|_\infty}{2N+1} = \frac{2|a|\|x\|_\infty}{2N+1} \tag{5}$$

Now we need to prove it is true if $a < 0$. It is easy to prove that T_a is an isometry. For any $a \in \mathbb{Z}$ and $x \in \ell^\infty$ we see that $\|T_a x\|_\infty = \sup_{n\in\mathbb{Z}}|x_{n+a}| = \sup_{n\in\mathbb{Z}}|x_n| = \|x\|_\infty$. If $a < 0$ we know that $-a > 0$, so

$$|M_n(x - T_a x)| = |-M_N(T_a x - x)| = |M_n(T_a x - T_{-a}T_a x)| \tag{6}$$
$$\leq \frac{2(-a)\|T_a x\|_\infty}{2N+1} = \frac{2|a|\|x\|_\infty}{2N+1} \tag{7}$$

in the last equality we used that T_a is an isometry.

(ii) We will first determine $\lim_{N\to\infty} M_n(e)$

$$\lim_{N\to\infty} M_n(e) = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{|n|\leq N} 1 = \lim_{N\to\infty}\frac{2N+1}{2N+1} = 1 \tag{8}$$

Because of (i) we know that for any $1 \leq j \leq p$

$$\lim_{N\to\infty}|M_N(f_j - T_{a_j} f_j| \leq \lim_{N\to\infty}\frac{2|a|}{2N+1}\|x\|_\infty = 0 \tag{9}$$

hence

$$\lambda_0 = \lambda_0 \cdot 1 + \sum_{j=1}^{p}\lambda_j \cdot 0 = \lambda_0 \cdot \lim_{N\to\infty} M_N(e) + \sum_{j=1}^{p}\lambda_j \cdot \lim_{N\to\infty} M_N(f_j - T_{a_j} f_j) \tag{10}$$
$$= \lim_{N\to\infty} M_N\left(\lambda_0 e + \sum_{j=1}^{p}\lambda_j(f_j - T_{a_j} f_j)\right) = \lim_{N\to\infty} M_N(x) \tag{11}$$

To show that $L \in V'$ we need to show that L is linear and bounded. In (iii) we will prove that $\|L\| = 1$, hence L is bounded and therefore we only need to show that L is linear. For this we need to prove that $L(x + \kappa x') = L(x) + \kappa L(x')$ for all $x, x' \in V$ and $\kappa \in \mathbb{F}$. We can write x and x' as $\lambda_0 e + \sum_{j=1}^{p}\lambda_j\left(f_j - T_{a_j} f_j\right)$ and $\lambda_0' e + \sum_{j=1}^{p'}\lambda_j'\left(f_j' - T_{a_j'} f_j'\right)$. Now define $\mu_0 = \lambda_0 + \kappa\lambda_0'$ and $\mu_i = \lambda_i, b_i = a_i, g_i = f_i$ for $1 \leq i \leq p$ and $\mu_i = \kappa\lambda_{i-p}', b_i = a_{i-p}', g_i = f_{i-p}$ for $p < i \leq p + p'$. We see that $x + \kappa y = \mu_0 e + \sum_{i=1}^{p+p'}\mu_i(g_i - T_{b_i})$, so $L(x + \kappa y) = \mu_0 = \lambda_0 + \kappa\lambda_0' = L(x) + \kappa L(x')$.

(iii) We saw in (ii) that $L(e) = 1$ and we know that $\|e\|_\infty = 1$, hence $\|L\| \geq 1$. For all $x \in V$ we see that

$$|L(x)| = |\lambda_0| = \lim_{N\to\infty} |M_N(x)| \leq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\leq N} |x_n| \leq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\leq N} \|x\|_\infty \tag{12}$$

$$= \lim_{N\to\infty} \frac{2N+1}{2N+1} \|x\|_\infty = \|x\|_\infty \tag{13}$$

hence $\|L\| = 1$.

(iv) We know that V is a linear subspace of $\ell^\infty(\mathbb{Z})$ and that $L \in V'$. Because of theorem 5.19 we know there exists $M \in (\ell^\infty)'$ such that $\|M\| = \|L\|$. Because it is an extension we know that $M(x) = L(x)$ for all $x \in V$, Also we see that $M(e) = L(e) = 1 = \|L\| = \|M\|$. Finally we also see that $x - T_a x \in V$ and also $L(x - T_a x) = 0$ for all $x \in \ell^\infty(\mathbb{Z})$ and $a \in \mathbb{Z}$. Thus

$$M(x) = M(x - T_a x) + M(T_a x) = L(x - T_a x) + M(T_a x) = M(T_a x)$$

(v) Assume that there exists an $y \in \ell^1(\mathbb{Z})$ such that

$$M(x) = \sum_{n\in\mathbb{Z}} x_n y_n$$

for all $(x_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$. We know $M(e) = 1$, therefore $\sum_{n\in\mathbb{Z}} y_n = 1$, so there must be a $n \in \mathbb{N}$ such that $y_n \neq 0$. Now define $e^n \in \ell^\infty(\mathbb{Z})$ as $e^n_n = 1$ and $e^n_i = 0$ for all $i \in \mathbb{Z}\backslash\{n\}$. We see because of (iv)(c) that $y_{n'} = M(e^{n'}) = M(T_{n'-n}e^n) = M(e^n) = y_n$ for all $n' \in \mathbb{Z}$. Therefore $(y_n)_{n\in\mathbb{Z}}$ is a constant sequence unequal to 0. Because of this $y \notin \ell^1(\mathbb{Z})$. This is a contradiction.

(vi) Define $S : \ell^\infty(\mathbb{Z}) \to (\ell^1(\mathbb{Z}))'$ as $S(x)(y) = \sum_{n\in\mathbb{Z}} x_n y_n$. We will show that S is well-defined and bijective. **well-defined:** Clearly $S(x)$ is linear for all $x \in \ell^\infty(\mathbb{Z})$. We see that

$$|S(x)(y)| = \left|\sum_{n\in\mathbb{Z}} x_n y_n\right| \leq \sum_{n\in\mathbb{Z}} |x_n||y_n| \leq \|x\|_\infty \sum_{n\in\mathbb{Z}} |y_n| = \|x\|_\infty \|y\|_1$$

, hence $\|S(x)\| \leq \|x\|_\infty$, also

$$\|S(x)\| = \sup_{\|y\|_1 = 1} |S(x)(y)| \geq \sup_{n\in\mathbb{Z}} |S(x)(e^n)| = \sup_{n\in\mathbb{Z}} |x_n| = \|x\|_\infty$$

and therefore $\|S(x)\| = \|x\|_\infty$, so $S(x) \in (\ell^1(\mathbb{Z}))'$ for all $x \in \ell^\infty(\mathbb{Z})$. **injective:** Let $x, x' \in \ell^\infty(\mathbb{Z})$ and assume $S(x) = S(x')$. We see that $x_n = S(x)(e^n) = S(x')(e^n) = x'_n$ for all $n \in \mathbb{Z}$, hence $x = x'$. **surjective:** Let $f \in \ell^1(\mathbb{Z})$. Choose $x_n := f(e^n)$ for all $n \in \mathbb{Z}$. We see that $\|x\|_\infty = \sup_{n\in\mathbb{Z}} |f(e^n)| \leq \sup_{n\in\mathbb{Z}} \|f\|\|e^n\| = \|f\|$, therefore $x \in \ell^\infty(\mathbb{Z})$. Also because f is continuous $S(x)(y) = \sum_{n\in\mathbb{Z}} x_n y_n = \sum_{n\in\mathbb{Z}} f(e^n) y_n = \sum_{n\in\mathbb{N}} f(y_n e^n) = \lim_{N\to\infty} \sum_{|n|\leq N} f(y_n e^n) = f(\lim_{N\to\infty} \sum_{|n|\leq N} y_n e^n) = f(y)$ for all $y \in \ell^1(\mathbb{Z})$.

Because of this we can define $\Psi : \left(\ell^1(\mathbb{Z})\right)' \to \mathbb{F}$ as $\Psi(S(x)) = M(x)$ for all $x \in \ell^\infty(\mathbb{Z})$. We show that Ψ is linear. Let $S(x), S(x') \in \left(\ell^1(\mathbb{Z})\right)'$ and $\lambda \in \mathbb{R}$ then

$$\Psi(S(x) + \lambda S(x')) = \Psi(S(x + \lambda x')) = M(x + \lambda x') = M(x) + \lambda M(x') = \Psi(S(x)) + \lambda \Psi(S(x'))$$

also it is bounded

$$|\Psi(S(x))| = |M(x)| \leq \|M\|\|x\|_\infty = \|M\|\|S(x)\|$$

hence $\|\Psi\| \leq \|M\|$ and therefore $\Psi \in \left(\ell^1(\mathbb{Z})\right)''$. Because of (v) we know that there is no $y \in \ell^1(\mathbb{Z})$ such that $\Psi(S(x)) = S(x)(y) = J_{\ell^1(\mathbb{Z})}(y)(S(x))$ for all $x \in \ell^\infty(\mathbb{Z})$, hence $\Psi \neq J_{\ell^1(\mathbb{Z})}(y)$ for all $y \in \ell^1(\mathbb{Z})$, hence $J_{\ell^1(\mathbb{Z})}(\ell^1(\mathbb{Z})) \neq \left(\ell^1(\mathbb{Z})\right)''$ and therefore $\ell^1(\mathbb{Z})$ is not reflexive.