Functional Analysis (WISB315)

Exercise sheet 8

Exercise 2 is a **hand-in exercise** ✍. The rules for the hand-ins are the same as always. Note however that this exercise sheet is particular: it is prepared in the form of a mock exam. The number of points for each question is just an indication for your own use. The intended duration is 3 hours. *During the actual exam, you can use one book of your choice. You can freely use the results from the lectures and the textbook by Rynne and Youngson (feel free to ask in case of doubts).*

Exercise 1. Let X be the subspace

$$X = \{x = (x_i)_{i\in\mathbb{N}} \in \ell^2 \,:\, x_i = 0 \text{ for all } i \text{ except for finitely many}\},$$

equipped with the induced ℓ^2 norm.

For each $n \in \mathbb{N}$, we define an operator $T_n : X \to \ell^2$ by

$$(T_n x)_i = \begin{cases} 0 \text{ if } i \neq n, \\ n x_n \text{ if } i = n. \end{cases}$$

(i) [0,5 p.] Show that for each $n \in \mathbb{N}$, $T_n \in B(X, \ell^2)$.

(ii) [0,5 p.] Show that for each $x \in X$, the sequence

$$A_x := (T_n x)_{n\in\mathbb{N}}$$

is bounded.

(iii) [0,5 p.] Let $A = (T_n)_{n\in\mathbb{N}} \subset B(X, \ell^2)$. Explain why the uniform boundedness principle (Banach–Steinhaus theorem) cannot be applied to deduce that A is bounded in $B(X, \ell^2)$.

(iv) [0,5 p.] Show that A is not bounded in $B(X, \ell^2)$.

In the next exercise, $\ell^1(\mathbb{Z})$ is the Banach space of all complex sequences $(x_n)_{n\in\mathbb{Z}}$ such that $\|x\|_1 < +\infty$, with norm $\|x\|_1 = \sum_{n\in\mathbb{Z}} |x_n|$, and $\ell^\infty(\mathbb{Z})$ is the Banach space of all complex sequences $(x_n)_{n\in\mathbb{Z}}$ such that $\|x\|_\infty < +\infty$, with norm $\|x\|_\infty = \sup_{n\in\mathbb{Z}} |x_n|$.

✍ **Exercise 2.** For each $N \in \mathbb{N}$ we define the functional $M_N : \ell^\infty(\mathbb{Z}) \to \mathbb{C}$ by

$$M_N(x) = \frac{1}{2N+1} \sum_{|n|\leq N} x_n, \quad \forall x \in \ell^\infty(\mathbb{Z}).$$

For each $a \in \mathbb{Z}$ we define the operator $T_a : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ by

$$(T_a x)_n = x_{n+a}, \quad \forall x \in \ell^\infty(\mathbb{Z}).$$

Let also $e \in \ell^\infty(\mathbb{Z})$ be defined by $e_n = 1$ for all $n \in \mathbb{Z}$, and let $V \subset \ell^\infty(\mathbb{Z})$ be the subspace of all x that can be written as a finite sum:

$$x = \lambda_0 e + \sum_{j=1}^{p} \lambda_j (f_j - T_{a_j} f_j), \tag{1}$$

for some (x-dependent) $p \in \mathbb{N}$, $\lambda_0, \ldots, \lambda_p \in \mathbb{C}$, $a_1, \ldots, a_p \in \mathbb{Z}$, and $f_1, \ldots, f_p \in \ell^\infty(\mathbb{Z})$.

(i) [1 p.] Show that for all $x \in \ell^\infty(\mathbb{Z})$ and $a \in \mathbb{Z}$,

$$|M_N(x - T_a x)| \leq \frac{2|a|}{2N+1} \|x\|_\infty .$$

(ii) [0,5 p.] For each $x \in V$, show that $\lim_{N\to\infty} M_N(x)$ equals the coefficient λ_0 in any decomposition (1). In consequence, we can define a map $L : V \to \mathbb{C}$ by

$$L(x) = \lambda_0.$$

Show that $L \in V'$.

(iii) [0,5 p.] Show that the functional L has norm $\|L\| = L(e) = 1$.
(iv) [1,5 p.] Show that there exists a functional $M \in (\ell^\infty(\mathbb{Z}))'$ such that:
  (a) $M(x) = L(x)$ for all $x \in V$,
  (b) $M(e) = \|M\| = 1$,
  (c) $M(x) = M(T_a x)$ for all $a \in \mathbb{Z}$ and $x \in \ell^\infty(\mathbb{Z})$.
(v) [0,5 p.] Show that there exists no $y \in \ell^1(\mathbb{Z})$ such that

$$M(x) = \sum_{n \in \mathbb{N}} x_n y_n, \quad \forall (x_n)_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}).$$

(vi) [0,5 p.] Using the result from the previous question show that $\ell^1(\mathbb{Z})$ is not reflexive.

In the next exercises, all vector spaces are over $\mathbb{C}$.

Exercise 3. Suppose $F \subset C([0,1])$ is a closed subspace with respect to the $L^2[0,1]$ norm.
(i) [0,5 p.] Show that F is closed in $C([0,1])$ equipped with the $\|\cdot\|_\infty$ norm.
(ii) [0,5 p.] Show that there exists $C > 0$ such that

$$\|f\|_2 \leq \|f\|_\infty \leq C\|f\|_2, \quad \forall f \in F.$$

(iii) [0,5 p.] Let $n \in \mathbb{N}$ and $(f_1, \ldots, f_n) \subset F$ an orthonormal set (for the $L^2[0,1]$ inner product). Show that for all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$,

$$\|\lambda_1 f_1 + \cdots + \lambda_n f_n\|_2^2 = |\lambda_1|^2 + \cdots + |\lambda_n|^2.$$

(iv) [0,5 p.] Using (ii), show that for all $t \in [0,1]$,

$$|\lambda_1 f_1(t) + \cdots + \lambda_n f_n(t)| \leq C(|\lambda_1|^2 + \cdots + |\lambda_n|^2)^{1/2}.$$

(v) [0,5 p.] By choosing $\lambda_1, \ldots, \lambda_n$ suitably, deduce that for all $t \in [0,1]$,

$$|f_1(t)|^2 + \cdots + |f_n(t)|^2 \leq C(|f_1(t)|^2 + \cdots + |f_n(t)|^2)^{1/2}.$$

(vi) [0,5 p.] Show that necessarily $n \leq C^2$. (In consequence, F is finite dimensional.)