**Exercise 9.** Let $\mathcal{H}$ be a real Hilbert space. Let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be bilinear and continuous, and let $\Lambda \geq 0$ be such that

$$|a(x, y)| \leq \Lambda \|x\| \|y\| \quad \forall x, y \in \mathcal{H}$$

Suppose that $a$ is coercive, i.e. there exists $\lambda > 0$ such that

$$a(x, x) \geq \lambda \|x\|^2 \quad \forall x \in \mathcal{H}$$

(a)

(i) Let $x \in \mathcal{H}$. Show that there exists a unique vector $z \in \mathcal{H}$ such that $a(x, y) = \langle z, y\rangle$ for all $y \in \mathcal{H}$

(ii) Define a map $A : \mathcal{H} \to \mathcal{H}$ by $x \mapsto z$ (where $z$ is as above), and show that $A \in B(H)$, with $\|A\| \leq \Lambda$.

(iii) Prove that $A$ is injective. (Hint: To do this you can estimate $\langle Ax, x\rangle$ using (2)).

(iv) Using Exercise 8, prove that $\operatorname{Im} A$ is closed.

(v) Show that $(\operatorname{Im} A)^{\perp} = \{0\}$. (Hint: For $x \in (\operatorname{Im} A)^{\perp}$, consider the inner product $\langle Ax, x\rangle$.)

(vi) Show that $A$ is surjective.

(vii) Show that $A^{-1} \in B(\mathcal{H})$, and prove $\left\|A^{-1}\right\| \leq \lambda^{-1}$.

---

(i) For this exercise we use the Riesz-Fréchet Theorem (Theorem 5.2). We know that $a(x, \cdot) : \mathcal{H} \to \mathbb{R}$ for every $x \in \mathcal{H}'$, hence $a(x, \cdot) \in \mathcal{H}'$. Because of theorem 5.2 we know there is a unique $z \in \mathcal{H}$ such that $a(x, y) = \langle y, z\rangle = \langle z, y\rangle$.

(ii) What is B(H)? Because of (i) $A$ is well-defined. We see that

$$\|Ax\|^2 = \langle Ax, Ax\rangle = a(x, Ax) \leq |a(x, Ax)| \leq \Lambda \|x\| \|Ax\| \tag{1}$$

for all $x \in \mathcal{H}$. If $Ax = 0$ it is trivial that $\|Ax\| = 0 \leq \Lambda\|x\|$ and if $Ax \neq 0$ we can divide both sides of (**??**) by $\|Ax\|$ and see that

$$\|Ax\| \leq \Lambda \|x\|$$

hence $\|A\| \leq \Lambda$

(iii) Let $x, x' \in \mathcal{H}$ and assume that $Ax = Ax'$. This means that $a(x, y) = \langle Ax, y\rangle = \langle Ax', y\rangle = a(x', y)$ for all $y \in \mathcal{H}$. Because of the bi-linearity of $a$ this means that $a(x - x', y) = a(x, y) - a(x', y) = 0$ for all $y \in \mathcal{H}$, hence $\lambda\|x - x'\|^2 \leq a(x - x', x - x') = 0$. Because $\Lambda > 0$ we know that $\|x - x'\|^2 = 0$, os $x = x'$.

(iv) For all $x \in \mathcal{H}$ we see that

$$\|Ax\| \|x\| \geq \langle Ax, x\rangle = a(x, x) \geq \lambda \|x\|^2 \tag{2}$$

If $x = 0$ we see that $\|Ax\| = 0 = c \cdot 0 = c\|x\|$. And otherwise we can divide (**??**) by $\|x\|$ and get that $\|Ax\| \geq \lambda\|x\|$ for all $x \in \mathcal{H}$. Also because $\mathcal{H}$ is a hilbert it must be a Banach space and in exercise (ii) we proved that $A \in B(\mathcal{H}, \mathcal{H})$. Beause of exercise 8 we know that Im$A$ is closed.

(v) We need to show that if $\langle Ax, y\rangle = 0$ for all $x \in \mathcal{H}$ that $y = 0$. Note that if $\langle Ax, y\rangle = 0$ for all $x \in \mathcal{H}$ that $\lambda\|y\|^2 \leq a(y, y) = \langle Ay, y\rangle = 0$, hence $\|y\| = 0$ and also $y = 0$.

(vi) Because $A$ is linear Im$A$ must also be linear. We know from (iv) that Im$A$ is closed, so $\text{Im}A = (\text{Im}A)^{\perp\perp}$ by corollary 3.35. We know from (v) that $\text{Im}A^{\perp} = \{0\}$, hence $\text{Im}A = (\text{Im}A)^{\perp\perp} = \{0\}^{\perp} = \mathcal{H}$, hence $A$ is surjective.

(vii) Because (iii) and (vi) we know that $A$ is injective and surjective, hence $A^{-1}$ is well-defined. Know we only need to show that it is bounded. By definition we know that $a(A^{-1}z, y) = \langle z, y \rangle$ for any $z, y \in \mathcal{H}$. We then see

$$\lambda \|A^{-1}z\|^2 \leq a(A^{-1}z, A^{-1}z) = \langle z, A^{-1}z \rangle \leq \|z\| \|A^{-1}z\| \tag{3}$$

If $A^{-1}z = 0$ we see that $\|A^{-1}z\| = 0 \leq \lambda \|z\|$ and otherwise we can divide both sides of (**??**) by $\lambda \|A^{-1}z\|$ and get that $|A^{-1}z\| \leq \lambda^{-1} \|z\|$.