Exercise 9. Let $\mathcal{H}$ be a real Hilbert space. Let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be bilinear and continuous, and let $\Lambda \geq 0$ be such that

$$|a(x, y)| \leq \Lambda \|x\| \|y\| \quad \forall x, y \in \mathcal{H}$$

Suppose that a is coercive, i.e. there exists $\lambda > 0$ such that

$$a(x, x) \geq \lambda \|x\|^2 \quad \forall x \in \mathcal{H}$$

(a)

(i) Let $x \in \mathcal{H}$. Show that there exists a unique vector $z \in \mathcal{H}$ such that $a(x, y) = \langle z, y\rangle$ for all $y \in \mathcal{H}$

(ii) Define a map $A : \mathcal{H} \to \mathcal{H}$ by $x \mapsto z$ (where z is as above), and show that $A \in B(H)$, with $\|A\| \leq \Lambda$.

(iii) Prove that A is injective. (Hint: To do this you can estimate $\langle Ax, x\rangle$ using (2)).

(iv) Using Exercise 8, prove that $\operatorname{Im} A$ is closed.

(v) Show that $(\operatorname{Im} A)^{\perp} = \{0\}$. (Hint: For $x \in (\operatorname{Im} A)^{\perp}$, consider the inner product $\langle Ax, x\rangle$.)

(vi) Show that A is surjective.

(vii) Show that $A^{-1} \in B(\mathcal{H})$, and prove $\left\|A^{-1}\right\| \leq \lambda^{-1}$.

(i) For this exercise we use the Riesz-Fréchet Theorem (Theorem 5.2). We know that a is bi-linear and continuous, hence $a(x, \cdot) : \mathcal{H} \to \mathbb{R}$ is linear and continuous for every $x \in \mathcal{H}$, hence $a(x, \cdot) \in \mathcal{H}'$. Because of theorem 5.2 we know there is a unique $z \in \mathcal{H}$ such that $a(x, y) = \langle y, z\rangle = \langle z, y\rangle$ for all $y \in \mathcal{H}$.

(ii) Because of (i) A is well-defined. We will also show it is linear. For this need to show that $A(\lambda x + x') = \lambda Ax + Ax'$ for all $x, x' \in \mathcal{H}$ and $\lambda \in \mathbb{R}$. We see that

$$a(\lambda x + x', y) = \langle A(\lambda x + x'), y\rangle$$

and that

$$a(\lambda x + x', y) = \lambda a(x, y) + a(x', y) = \lambda\langle Ax, y\rangle + \langle Ax', y\rangle = \langle \lambda Ax + Ax', y\rangle$$

for all $y \in \mathcal{H}$. Because of (i) we know there is a unique $z \in \mathcal{H}$ such that $a(\lambda x + x', y) = \langle z, y\rangle$ for all $y \in \mathcal{H}$, hence $A(\lambda x + x') = \lambda Ax + Ax'$

Now we will show that A is bounded. We see that

$$\|Ax\|^2 = \langle Ax, Ax\rangle = a(x, Ax) \leq |a(x, Ax)| \leq \Lambda \|x\| \|Ax\| \tag{1}$$

for all $x \in \mathcal{H}$. If $Ax = 0$ it is trivial that $\|Ax\| = 0 \leq \Lambda\|x\|$ and if $Ax \neq 0$ we can divide both sides of (1) by $\|Ax\|$ and see that

$$\|Ax\| \leq \Lambda \|x\|$$

hence $\|A\| \leq \Lambda$ and $A \in B(H)$.

(iii) Let $x, x' \in \mathcal{H}$ and assume that $Ax = Ax'$. This means that $a(x, y) = \langle Ax, y\rangle = \langle Ax', y\rangle = a(x', y)$ for all $y \in \mathcal{H}$. Because of the bi-linearity of a this means that $a(x - x', y) = a(x, y) - a(x', y) = 0$ for all $y \in \mathcal{H}$, hence $\lambda\|x - x'\|^2 \leq a(x - x', x - x') = 0$. Because $\lambda > 0$ we know that $\|x - x'\|^2 = 0$, so $x = x'$.

(iv) For all $x \in \mathcal{H}$ we see that

$$\|Ax\|\|x\| \geq \langle Ax, x\rangle = a(x,x) \geq \lambda\|x\|^2 \tag{2}$$

If $x = 0$ we see that $\|Ax\| = 0 = c \cdot 0 = c\|x\|$. And otherwise we can divide (2) by $\|x\|$ and get that $\|Ax\| \geq \lambda\|x\|$ for all $x \in \mathcal{H}$. Also because $\mathcal{H}$ is a hilbert it must be a Banach space and in exercise (ii) we proved that $A \in B(\mathcal{H}, \mathcal{H})$. Beause of exercise 8 we know that ImA is closed.

(v) We need to show that if $\langle z, y\rangle = 0$ for all $z \in \mathrm{im}(A)$ that $y = 0$. In other words we need to show that if $\langle Ax, y\rangle = 0$ for all $x \in \mathcal{H}$ that $y = 0$. Note that if $\langle Ax, y\rangle = 0$ for all $x \in \mathcal{H}$ that $\lambda\|y\|^2 \leq a(y,y) = \langle Ay, y\rangle = 0$, hence $\|y\| = 0$ and also $y = 0$.

(vi) Because of (ii) we know that A is linear and therefore ImA must be a linear space. We know from (iv) that ImA is closed, so $\mathrm{Im}A = (\mathrm{Im}A)^{\perp\perp}$ by corollary 3.35. We know from (v) that $\mathrm{Im}A^{\perp} = \{0\}$, hence $\mathrm{Im}A = (\mathrm{Im}A)^{\perp\perp} = \{0\}^{\perp} = \mathcal{H}$, hence A is surjective.

(vii) Because (iii) and (vi) we know that A is injective and surjective, hence A^{-1} is well-defined. We know that the inverse of a linear map is linear, hence A^{-1} is linear. Now we only need to show that it is bounded. By definition we know that $a(A^{-1}z, y) = \langle z, y\rangle$ for any $z, y \in \mathcal{H}$. We then see

$$\lambda\|A^{-1}z\|^2 \leq a(A^{-1}z, A^{-1}z) = \langle z, A^{-1}z\rangle \leq \|z\|\|A^{-1}z\| \tag{3}$$

If $A^{-1}z = 0$ we see that $\|A^{-1}z\| = 0 \leq \lambda^{-1}\|z\|$ and otherwise we can divide both sides of (3) by $\lambda\|A^{-1}z\|$ and get that $|A^{-1}z\| \leq \lambda^{-1}\|z\|$.