Analysis
Solution for Week 3

February 28, 2024

Exercise 2.18 [9 points]
Let (V, d) be a metric space. Define $\tilde{d}\colon V \times V \to \mathbb{R}$ by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in V.$$

Show that $(V, \tilde{d})$ is a metric space.

Hint: To prove the triangle inequality, show that for $a, b, c \geq 0$, the inequality $c \leq a + b$ implies that

$$\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

Solution:
We have to show that $\tilde{d}$ is a metric on V, i.e., it satisfies the three properties of Definition 2.11. In the following, let $x, y, z \in V$ be arbitrary.

(a) Using $d(x, y) \geq 0$ immediately yields

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \geq 0.$$

Further, $\tilde{d}(x, y) = 0$ if and only if $d(x, y) = 0$ if and only if $x = y$.

(b) Using $d(x, y) = d(y, x)$ yields

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = \tilde{d}(y, x).$$

(c) We make the following observation: Let $a, b, c \geq 0$ such that $c \leq a + b$. Adding $c(a + b)$ to both sides of the inequality and rearranging yields

$$\frac{c}{1+c} \leq \frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b},$$

where in the last step we dropped $b \geq 0$ and $a \geq 0$, respectively. (Note that it is important that the denominator is at least 1.)

For $c \coloneqq d(x, z)$, $a \coloneqq d(x, y)$, and $b \coloneqq d(y, z)$, it holds $c \leq a + b$ by the triangle inequality of d. Applying the above observation then yields the triangle inequality for $\tilde{d}$.

Correction guidelines: part (a) (2 points); part (b) (1 point); part (c) proving the hint (3 points) and applying it (1 point); style (2 points). The grade is calculated as number of points + 1.