ANALYSIS WEEK 1 SOLUTION

Point division is given at the end of the file.

1. Exercise 1.34

Part a)

Problem 1.1. Show that for all $x, y \in \mathbb{R}^n$,

$$(1 + \|x + y\|) \leq (1 + \|x\|)(1 + \|y\|). \tag{1.1}$$

Proof. Observe that by the triangle inequality, $\|x+y\| \leq \|x\|+\|y\|$ for all $x, y \in \mathbb{R}^n$. Adding 1 to both sides, we obtain

$$1 + \|x + y\| \leq 1 + \|x\| + \|y\|.$$

Next, we observe that $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$, meaning that in particular, $\|x\|\|y\| \geq 0$ for all $x, y \in \mathbb{R}^n$. Therefore, we conclude that

$$1 + \|x + y\| \leq 1 + \|x\| + \|y\| \leq 1 + \|x\| + \|y\| + \|x\|\|y\| = (1 + \|x\|)(1 + \|y\|). \tag{1.2}$$

□

Part b)

Problem 1.2. Show that for all $x, y \in \mathbb{R}^n$

$$1 + \|x + y\| \geq \frac{1 + \|x\|}{1 + \|y\|}. \tag{1.3}$$

(Hint: Use part a) and that $\|y\| = \| - y\|$.

Proof. Observe that it is equivalent to prove

$$(1 + \|x\|) \leq (1 + \|y\|)(1 + \|x + y\|) \tag{1.4}$$

for all $x, y \in \mathbb{R}^n$. Next, we use that $x = x + y - y$. Applying part a), we see that

$$1 + \|x\| = 1 + \|x + y - y\| \leq (1 + \|x + y\|)(1 + \| - y\|). \tag{1.5}$$

The proof is concluded by noting that $\| - y\| = \|y\|$. □

2. Point division

Total number of points that can be earned is 10 points. 2 points can be awarded for style of the proof. The remaining points are divided equally over the two parts as follows:

- Correct use of the triangle inequality in part a). (*2 points*)
- Observing that $\|x\|\|y\| \geq 0$ to conclude the proof of part a). (*2 points*)
- Observing that it is equivalent to prove (1.4). (*1 point*)
- Using the $x = x + y - y$ and applying part a) correctly. (*3 points*)