Assignment 4.2
Jeroen Meringa
0281913
13-03-2024

(a) Let $(b_n)_{n\in\mathbb{N}}$ be a converging sequence in metric space (V,d). Show that $(b_n)_{n\in\mathbb{N}}$ is a Cauchy sequece, meaning that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ it is true that $d(b_n, b_m) < \epsilon$.

Proof:

As $(b_n)_{n\in\mathbb{N}}$ is a converging sequence, it converges to some limit b. Let $\epsilon > 0$. Then there exists an integer N such that $d(b_m, b) < \frac{\epsilon}{2}$ and $d(b_n, b) < \frac{\epsilon}{2}$ for all $m, n \geq N$. Invoking the triangle inequality, we can conclude that $d(b_m, b_n) \leq d(b_m, b) + d(b, b_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $m, n \geq N$. This shows that the sequence is Cauchy.

Q.E.D.

(b) Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a metric spacce V. Show that there exists an $N \in \mathbb{N}$ such that $a_n \in B(a_N; 1)$ for all $n \geq N$.

Proof:

Consider the case where $\epsilon = 1$. By the Cauchy property, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $d(a_n, a_m) < 1$. Now consider $m = N$. Then $d(a_N, a_n) < 1$ for all $n \geq N$ implying that a_n is within the open ball of radius 1 around a_N, and therefore $a_n \in B(a_N; 1)$ for all $n \geq N$.

Q.E.D.

(c) Let $(a_n)_{n\in\mathbb{N}}$ as before. Show that there exist a $b \in V$ and $R > 0$ such that $a_n \in B(b; R)$ for all $n \in \mathbb{N}$.

Proof:

Let $R = \max_{0\leq n<N} d(x_n, x_N) + 1$ be the maximum distance for any a_n to a_N for $0 \leq n < N$, and $b = a_N$. Then $d(a_n, b) < R$ and $a_n \in B(b; R)$ for all $0 \leq n < N$. Since $R \geq 1$,we've established in part (b) that $a_n \in B(b; R)$ for all $n \geq N$. Therefore, since $a_n \in B(b; R)$ for all $n < N$ and all $n \geq N$, $a_n \in B(b; R)$ for all $n \in \mathbb{N}$.

Q.E.D.

(d) Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^p$ with $p \geq 1$. Show that $(a_n)_{n\in\mathbb{N}}$ has a converging subsequence $(a_{n_j})_{j\geq 1}$.

Proof:

We show this by proving that the Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ is bounded, and then invoke the Bolzano-Weierstrass theorem.
Given that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $d(a_n, a_m) < \epsilon$ for all $\epsilon > 0$. Now, consider the set $\{a_1, a_2, \ldots, a_{N-1}\}$ consisting of the first $N-1$ terms of the sequence. Since there are only finitely many terms, it is a finite set. Let M be the maximum distance from the origin to any point in this set: $M = \max(||a_1||, ||a_2||, \ldots, ||a_{N-1}||)$. Then $a_{n<N} \in B(0; M)$ and the sequence $(a_n)_{n<N}$ is bounded by a sphere with radius M.
Now, let $\epsilon = 1$ and $a_m = a_N$. For any $n \geq N$, we then have that $||a_n - a_N|| < 1$ for all $n \geq N$ by the Cauchy property. Invoking the triangle inequality, we can write: $||a_n|| \leq ||a_n - a_N|| + ||a_N|| < 1 + ||a_N||$ implying that for all $n \geq N$, the norm of each a_n is bounded by $1 + ||a_N||$.
Since $(a_n)_{n\in\mathbb{N}}$ is thus bounded for all $n < N$ and for all $n \geq N$, it is a bounded sequence. Using the Bolzano-Weierstrass theorem, it therefore has a converging subsequence $(a_{n_j})_{j\geq 1}$.

Q.E.D.

(e) Let $a \in \mathbb{R}^p$ be the limit of the previous subsequence $(a_{n_j})_{j\geq 1}$. Prove that $\lim_{n\to\infty} a_n = a$.

Proof:

Let $\epsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ is Cauchy, there exists an $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$, we have $||a_n - a_m|| < \frac{\epsilon}{2}$. Additionally, since $(a_{n_j})_{j\geq 1}$ is a converging subsequence with limit a, there exists an $N_2 \in \mathbb{N}$ such that for all $j \geq N_2$, we have $||a_{n_j} - a|| < \frac{\epsilon}{2}$. Now, let $N = \max(N_1, N_2)$. For any $n \geq N$ consider the terms a_n and a_{n_j}. Using the triangle inequality, then $||a_n - a|| \leq ||a_n - a_m|| + ||a_{n_j} - a|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This shows that for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $||a_n - a|| < \epsilon$ which is the definition of the limit, proving that $\lim_{n\to\infty} a_n = a$.

Q.E.D.

(f) Conclude that $\mathbb{R}$ is complete.

Proof:

Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. In (d), we have shown that this sequence is bounded and has a convergent subsequence that is bounded as well. In (e), we then showed that this means that the original sequence $(a_n)_{n\in\mathbb{N}}$ converges to a limit in $\mathbb{R}$ itself as well. This means that every Cauchy sequence in $\mathbb{R}$ converges to a limit in $\mathbb{R}$, defining that $\mathbb{R}$ is complete.

Q.E.D.