# Exercise: Formulate integer programming problems

## 1 Linear programming problem properties

Find necessary and sufficient conditions for the numbers $s$ and $t$ to make the linear programming problem

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & s \cdot x_1 + t \cdot x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(i) has an optimal solution,

(ii) be infeasible,

(iii) be unbounded.

(i) $s > 0$ and $t > 0$

(ii) Never. (Instead, if we change the constraint into $s \cdot x_1 + t \cdot x_2 \leq -1$, the linear programming problem is infeasible.)

(iii) $s \leq 0$ and/or $t \leq 0$

## 2 LP formulation

A meat packing plant produces 480 hams, 400 pork bellies, and 230 picnic hams every day; each of these products can be sold either fresh or smoked. The total number of hams, bellies, and picnics that can be smoked during a normal working day is 420; in addition, up to 250 products can be smoked on overtime at a higher cost. The *net* profit are as follows.

| | Fresh | Smoked on regular time | Smoked on overtime |
|---|---|---|---|
| Hams | \$8 | \$14 | \$11 |
| Bellies | \$4 | \$12 | \$7 |
| Picnics | \$4 | \$13 | \$9 |

The objective is to find the schedule that maximizes the total net profit. Formulate as an LP problem.

Let $h_f, h_{sr},$ and $h_{so}$ be the amount of fresh, smoked on regular time, and smoked over time ham, respectively. The variables $b_f, b_{sr}, b_{so}, p_f, p_{sr},$ and $p_{so}$ are defined for bellies and picnics, similarly.

$$
\begin{aligned}
\text{maximize} \quad & 8h_f + 14h_{sr} + 11h_{so} + 4b_f + 12b_{sr} + 7b_{so} + 4p_f + 13p_{sr} + 9p_{so} \\
\text{subject to} \quad & h_f + h_{sr} + h_{so} \leq 480 \\
& b_f + b_{sr} + b_{so} \leq 400 \\
& p_f + p_{sr} + p_{so} \leq 230 \\
& h_{sr} + b_{sr} + p_{sr} \leq 420 \\
& h_{so} + b_{so} + p_{so} \leq 250 \\
& h_f, h_{sr}, h_{so}, b_f, b_{sr}, b_{so}, p_f, p_{sr}, p_{so} \geq 0
\end{aligned}
$$

## 3 Variation of the knapsack problem

Suppose that you are interested in choosing a set of investments $\{1, \cdots, 7\}$ using $0-1$ variables. Model the following constraints:

(i) You must choose at least one of them.

(ii) You cannot invest in all of them.

(i) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 1$

(ii) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 6$

## 4 Simplex method

Consider the linear programming problem:

$$
\begin{array}{lcccl}
\text{maximize} & 40x & + & 30y & \\
\text{subject to} & x & + & 2y & \leq 16 \\
 & x & + & y & \leq 9 \\
 & 3x & + & 2y & \leq 24 \\
 & & & x, y & \geq 0
\end{array}
$$

Recall that during the lecture, we started at the feasible solution $(x, y) = (0, 0)$, and then improved the objective value by fixing $x = 0$ and increasing $y$. Please run the simplex method by fixing $y = 0$ and increasing $x$.

**Round 0: adding slack variables.**

$$
\begin{array}{lrl}
\text{maximize} & 40x + 30y & \\
\text{subject to} & s_1 = & 16 - x - 2y \\
 & s_2 = & 9 - x - y \\
 & s_3 = & 24 - 3x - 2y \\
 & x, y, s_1, s_2, s_3 \geq & 0
\end{array}
$$

**Round 1: fixing $y = 0$ and increasing $x$.** Since $s_1, s_2, s_3 \geq 0$, the value of $x$ cannot be larger than $\min\{16, 9, 8\} = 8$. By setting $x = 8$, we get an objective value of $40 \cdot 8 + 0 = 320$.

In the case of $x = 8$ and $y = 0$, the constraint $s_3 = 24 - 3x - 2y$ is tight. We replace all $x$ by $8 - \frac{2}{3}y - \frac{1}{3}s_3$ and get the following linear programming problem:

$$\begin{aligned}
\text{maximize} \quad & 40(8 - \frac{2}{3}y - \frac{1}{3}s_3) + 30y \\
\text{subject to} \quad & s_1 = 16 - (8 - \frac{2}{3}y - \frac{1}{3}s_3) - 2y \\
& s_2 = 9 - (8 - \frac{2}{3}y - \frac{1}{3}s_3) - y \\
& s_3 = 24 - 3(8 - \frac{2}{3}y - \frac{1}{3}s_3) - 2y \\
& x, y, s_1, s_2, s_3 \geq 0
\end{aligned}$$

That is:

$$\begin{aligned}
\text{maximize} \quad & 320 + \frac{10}{3}y - 20s_3 \\
\text{subject to} \quad & s_1 = 8 - \frac{4}{3}y + \frac{1}{3}s_3 \\
& s_2 = 1 - \frac{1}{3}y + \frac{1}{3}s_3 \\
& x = 8 - \frac{2}{3}y - \frac{1}{3}s_3 \\
& x, y, s_1, s_2, s_3 \geq 0
\end{aligned}$$

**Round 2: Since the only variable in the objective with a positive coefficient is $y$, we fix $s_3 = 0$ and increase $y$.** Since $s_1, x \geq 0$, the value of $y$ cannot be larger than $\min\{12, 3\} = 3$. By setting $y = 3$, we get an objective value of $320 + \frac{10}{3} \cdot 3 + 0 = 330$.

In the case of $y = 3$ and $s_3 = 0$, the constraint $s_2 = 1 - \frac{1}{3}y + \frac{1}{3}s_3$ is tight. We replace all $y$ by $3 - 3s_2 + s_3$ and get the following linear programming problem:

$$\begin{aligned}
\text{maximize} \quad & 320 + \frac{10}{3}(3 - 3s_2 + s_3) - 20s_3 \\
\text{subject to} \quad & s_1 = 8 - \frac{4}{3}(3 - 3s_2 + s_3) + \frac{1}{3}s_3 \\
& y = 3 - 3s_2 + s_3 \\
& x = 8 - \frac{2}{3}(3 - 3s_2 + s_3) - \frac{1}{3}s_3 \\
& x, y, s_1, s_2, s_3 \geq 0
\end{aligned}$$

That is:

$$\begin{aligned}
\text{maximize} \quad & 330 - 10s_2 - \frac{50}{3}s_3 \\
\text{subject to} \quad & s_1 = 4 + 4s_2 - s_3 \\
& y = 3 - 3s_2 + s_3 \\
& x = 6 + 2s_2 - s_3 \\
& x, y, s_1, s_2, s_3 \geq 0
\end{aligned}$$

After round 2, the objective value 330 cannot be further improved by increasing $s_2$ or $s_3$ since $s_2, s_3 \geq 0$. Therefore, 330 is the optimal value.