Exercise 1: Ever Given

1 Ever Given

In March 2021, a container ship *Ever Given* blocked Suez Canal. Consider your are the captain of one of the ships that is waiting for passing through Suez Canal. You want to arrive your destination as soon as possible. However, because of the obstruction, your ship is idle near by Suez Canal and you have no idea when the canal will be available again. You have two options:

- to wait until the canal is available, and you can pass through the canal in F unit of time; or
- to go around Africa via the Cape of Good Hope, and you can arrive your destination in S units of time.

Note that F stands for fast, S stands for slow, and $F < S$.

Answer the following questions:

1. What is an *instance* in this problem?
2. What is the *cost* in this problem?
3. Consider the following strategy ALG_{S-F} that keeps waiting until the $(S-F)$-th time unit:

Algorithm 1 ALG_{S-F}

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t ← the current time
while t < S − F do
    if The canal is available then
        Pass through the canal
    else
        Wait
Turn around and take the Cape of Good Hope route    ▷ the canal is still blocked at the S − F-th unit of time
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   (a) Prove that ALG_{S-F} is $(2 - \frac{F}{S})$-competitive.
   (Hint:
      i. Let a be the actual time when the canal is available. How much is the optimal cost when $a < S - F$? How much is the cost of ALG_{S-F}?
      ii. Let a be the actual time when the canal is available. How much is the optimal cost when $a \geq S - F$? How much is the cost of ALG_{S-F}?
)
   (b) Show that your analysis is tight.

2 Buy on the $\frac{B}{2}$-th day[1]

Recall the Ski Rental problem mentioned in the lecture. Now, assume that $B \geq 1$ is even and consider the algorithm $\text{ALG}_{\frac{B}{2}}$ which buys the ski on the $\frac{B}{2}$-th day. Answer the following questions:

1. Prove that $\text{ALG}_{\frac{B}{2}}$ is at least $(3 - \frac{2}{B})$-competitive.

2. Prove that $\text{ALG}_{\frac{B}{2}}$ is $(3 - \frac{2}{B})$-competitive.
   (Hint:
   Consider the case where the number of skiing days $d < \frac{B}{2}$, $d \geq B$, and $\frac{B}{2} \leq d < B$. The last case is the most tricky one, where the algorithm buys the ski but the optimal keeps renting the ski. The ratio between the algorithm cost and the optimal cost in this case is $\frac{\frac{B}{2}-1+B}{d}$. You need to find a good value of d so you can get rid of it from the ratio while still having a valid upper bound.
)

3 Two-store Ski-Rental

Recall the Ski Rental problem mentioned in the lecture. Now, consider that there are two stores, 1 and 2, where you can buy or rent a pair of skis. Let r_1 and B_1 be the renting and buying prices from Store 1, respectively. For Store 2, r_2 and B_2 are defined symmetrically. Note that there is no specific relation between r_1 and r_2 or B_1 and B_2, except that B_1 and B_2 are both larger than $\max\{r_1, r_2\}$. Design a 2-competitive algorithm and show the competitiveness.

[1] Also try buying on the $2B$-th day.