Algorithms for Decision Support

NP-Completeness (1/3)

Turing Machine, **P**, and **NP**

Overview

- Algorithms:Turing machine, Deterministic and non-deterministic
- Formal language framework: string and language
- Time complexity
  - Input size
  - Classes **P** and **NP**
    - Polynomial time verification

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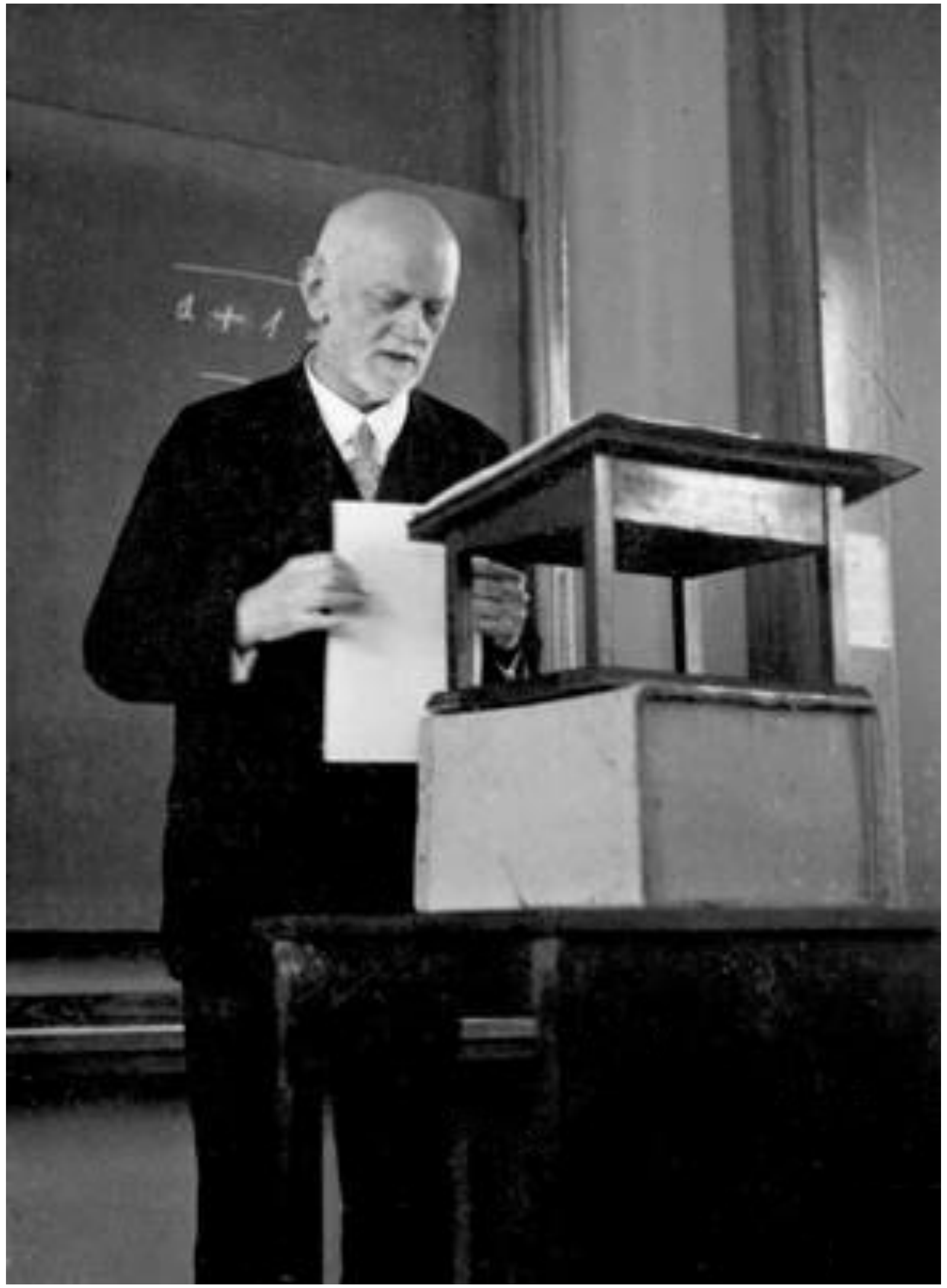

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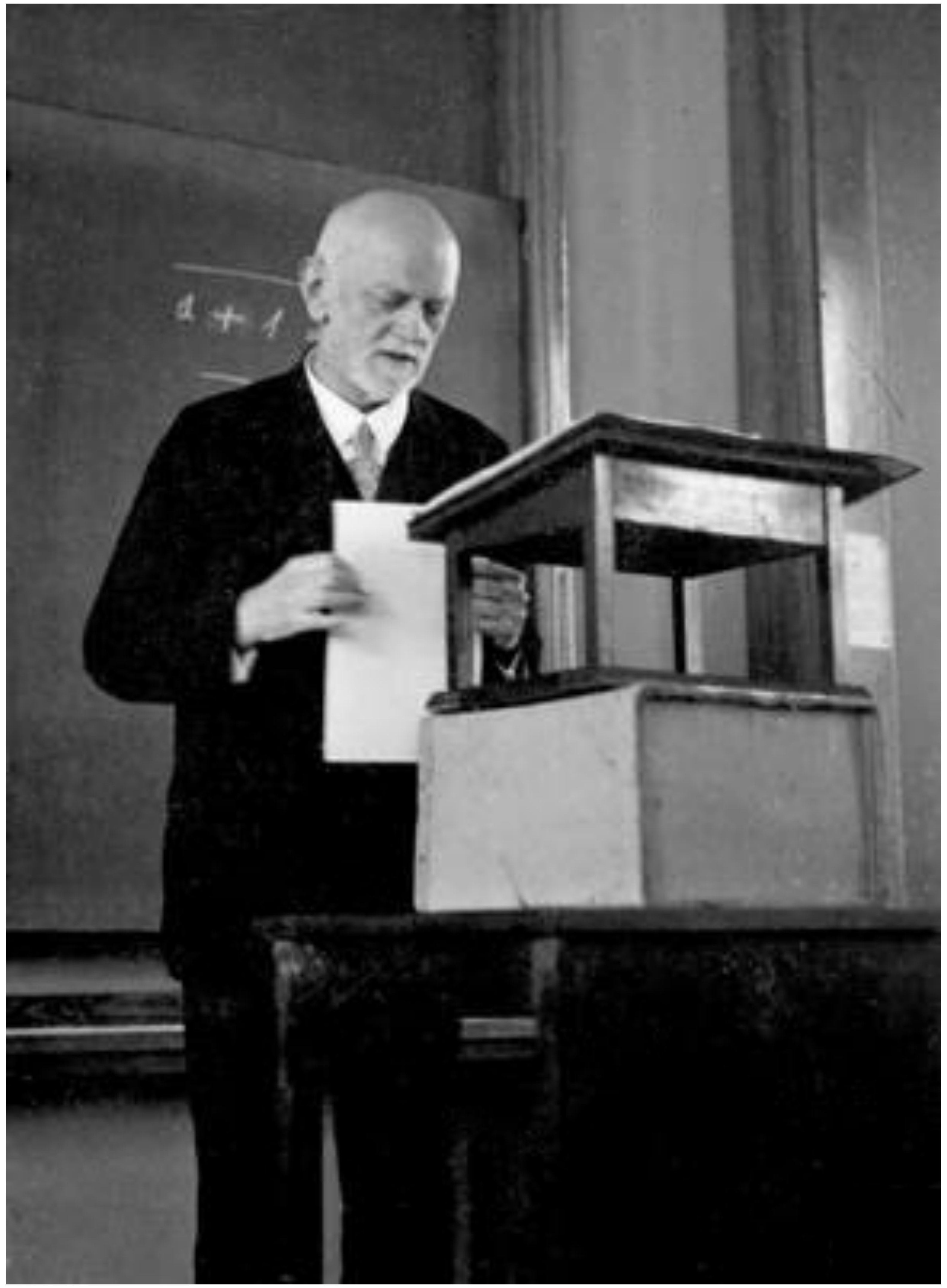

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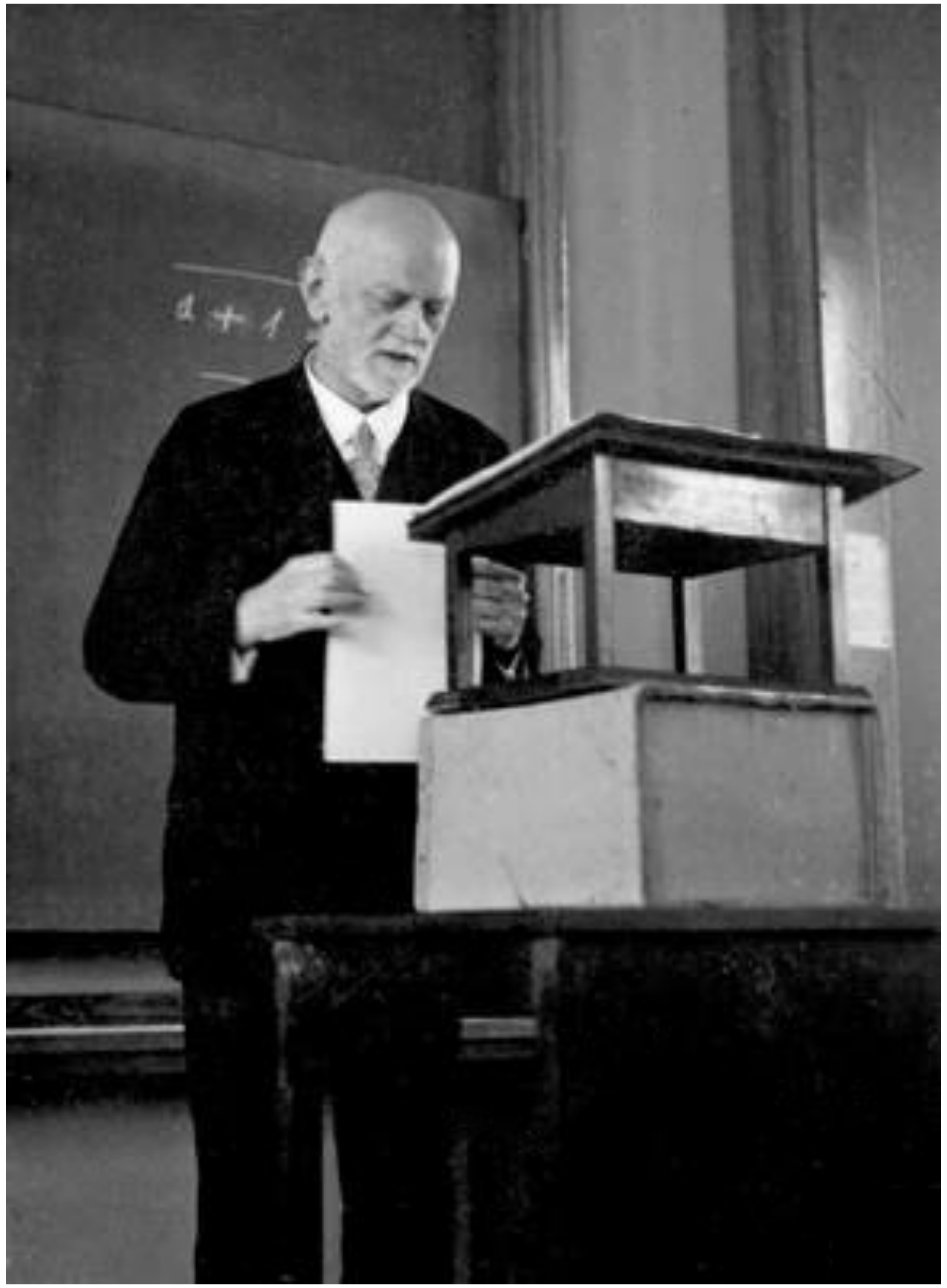

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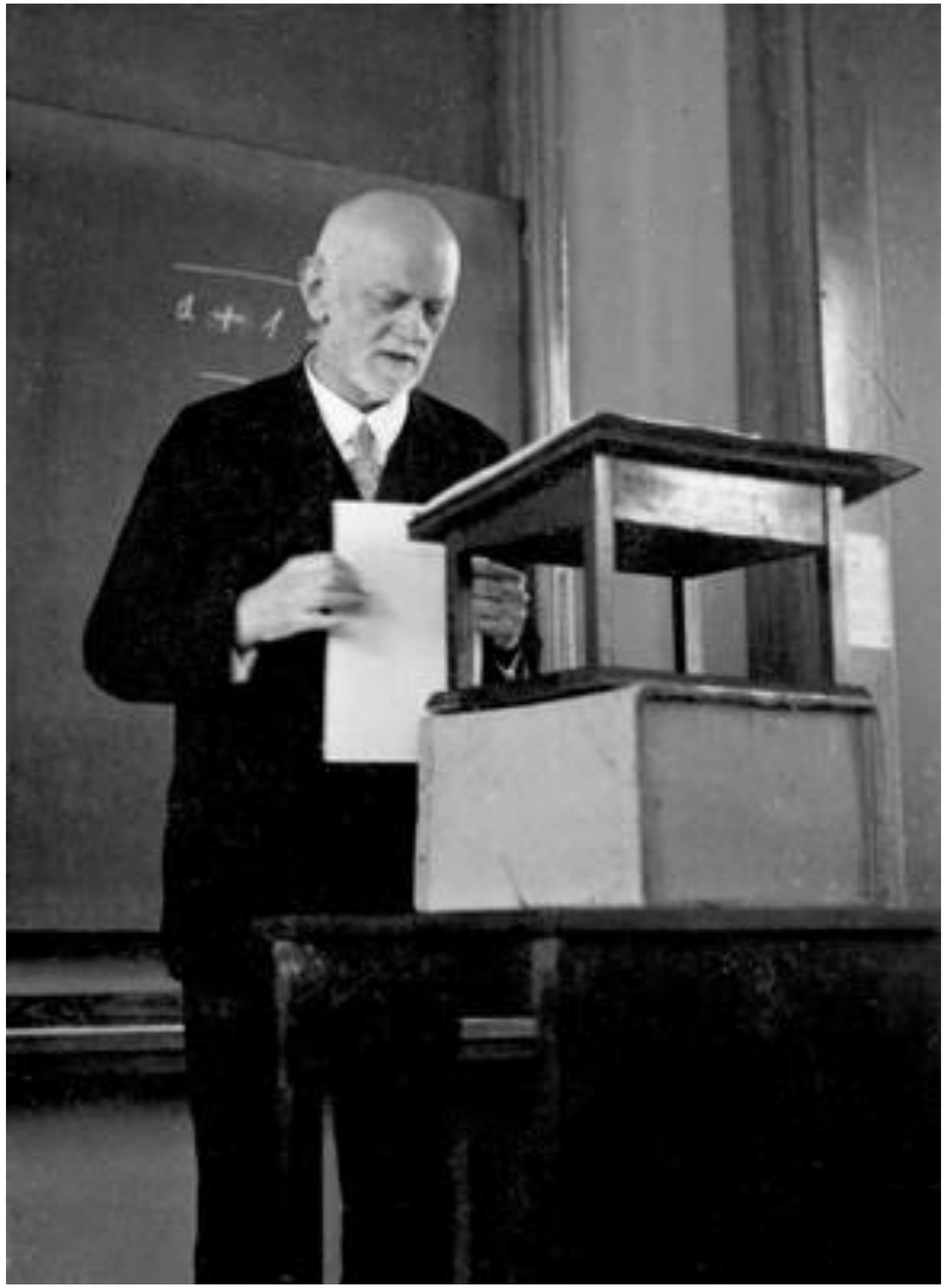

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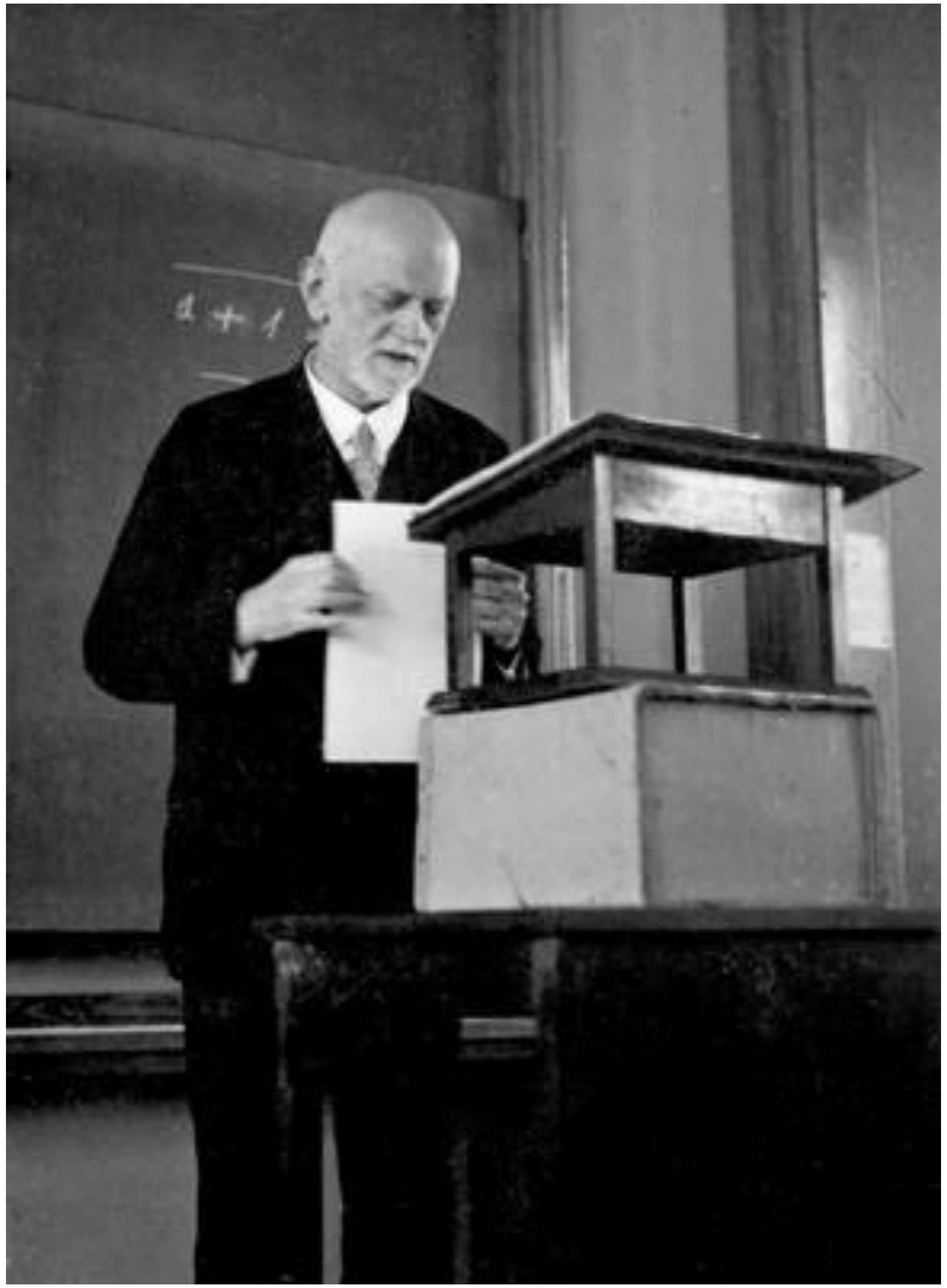

What is an algorithm/computer?

Turing Machine

- Proposed by Alan Turing in 1936

Alan Turing 1912~1954

Turing Machine

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- A model of a general purpose computer:
  a Turing machine can do every thing that a real computer can do

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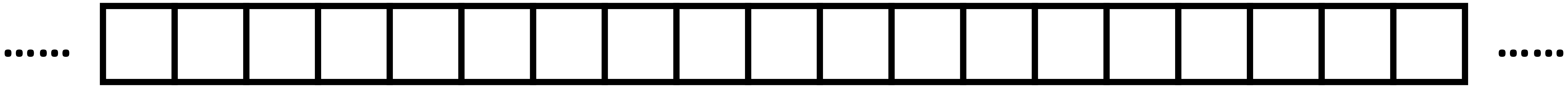

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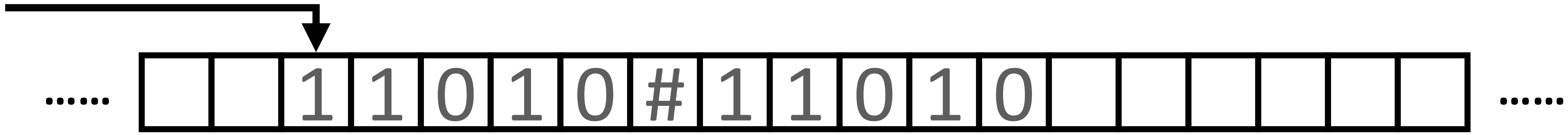

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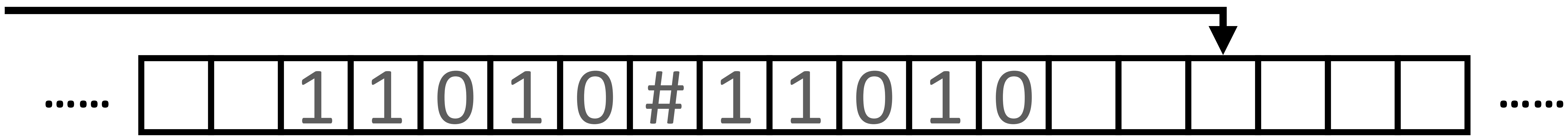

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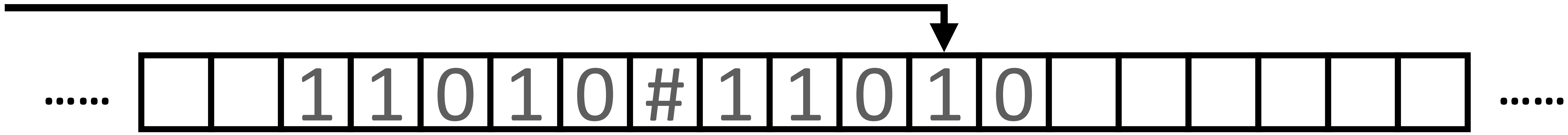

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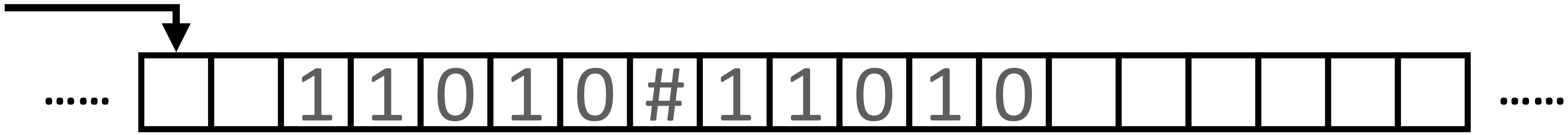

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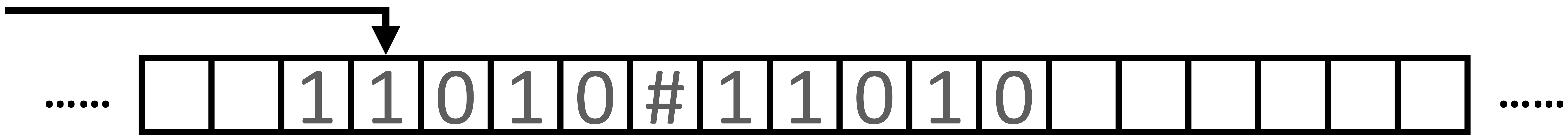

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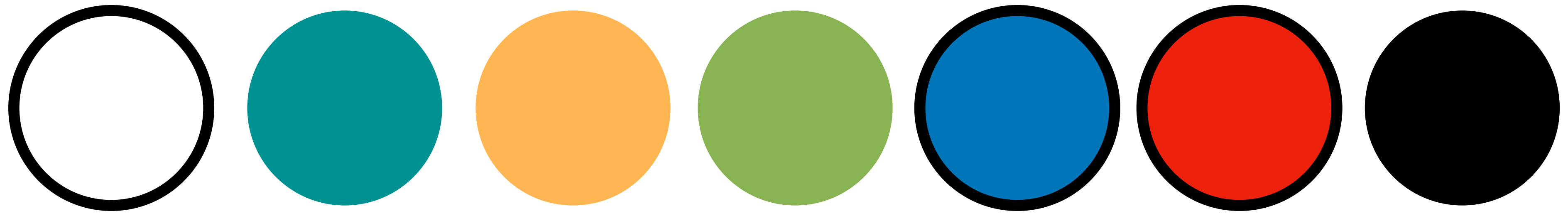

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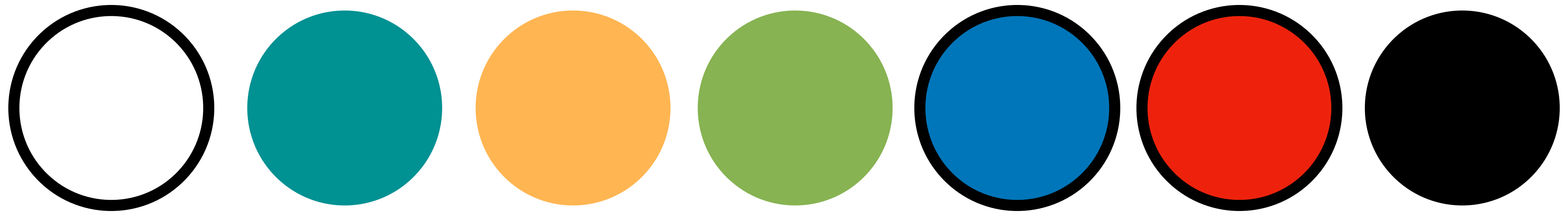

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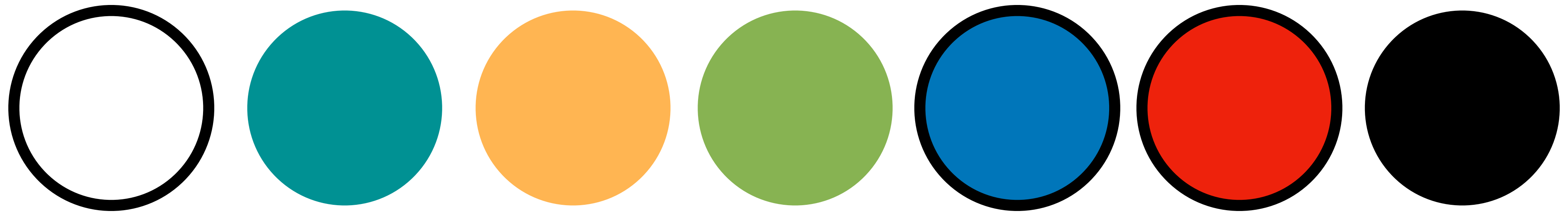
  - Finite-state control 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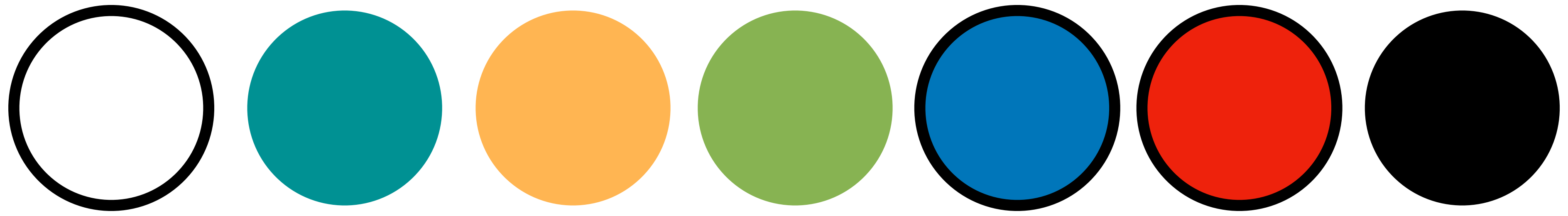

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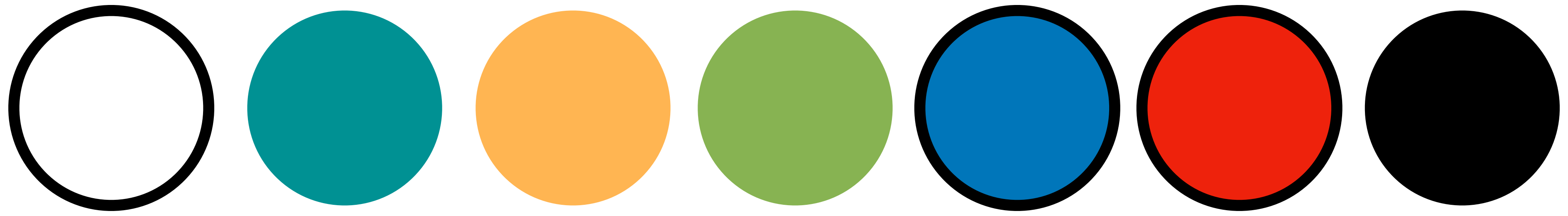
  - Finite-state control 

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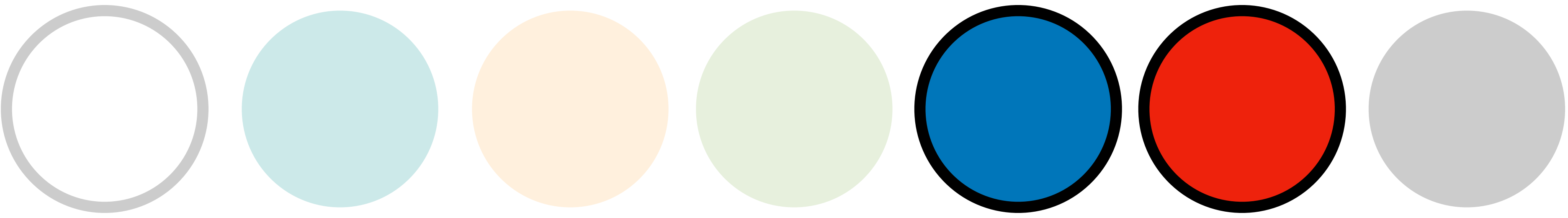
  - Finite-state control 
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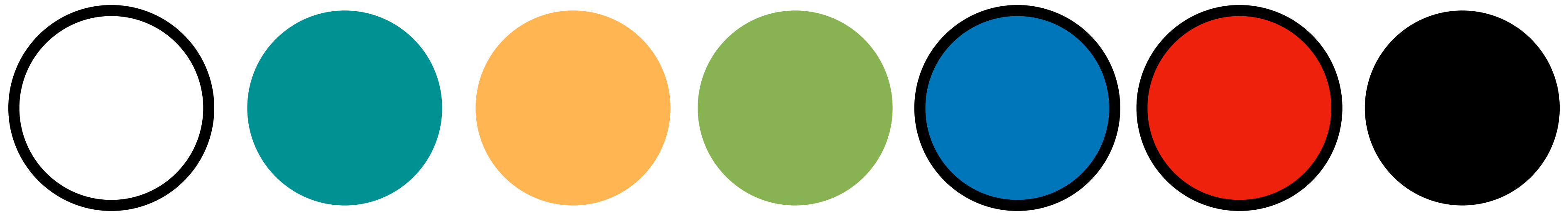
  - Finite-state control 
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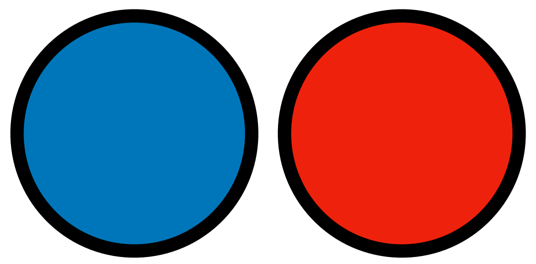

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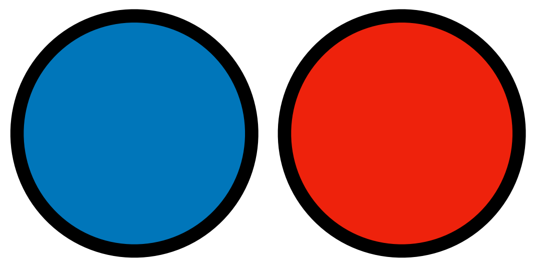

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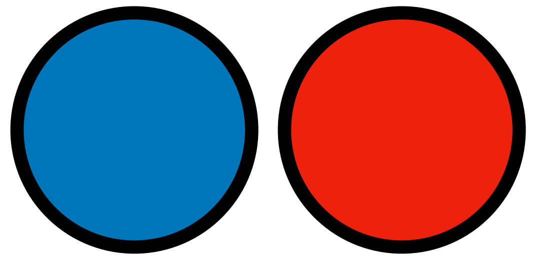

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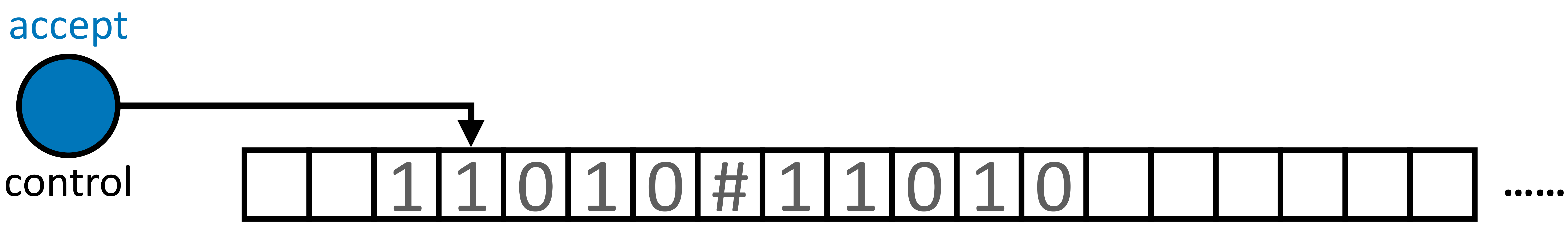

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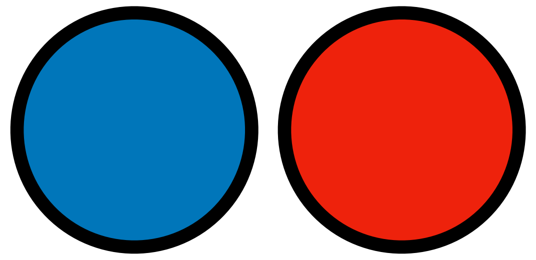
- Output: ***accept*** or ***reject*** (both **halting** configurations) 
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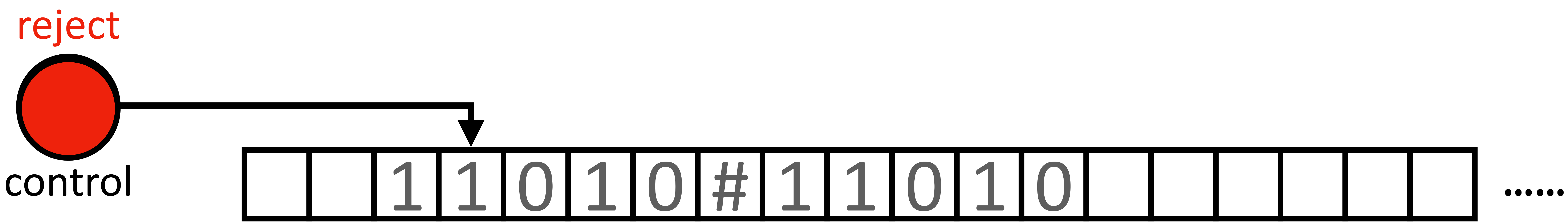

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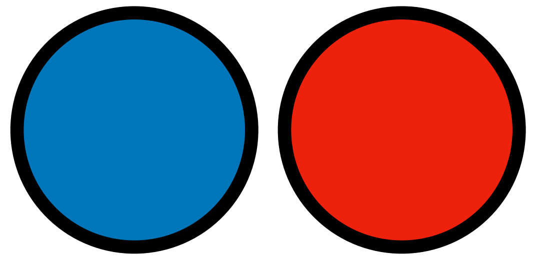
- Output: ***accept*** or ***reject*** (both **halting** configurations) 
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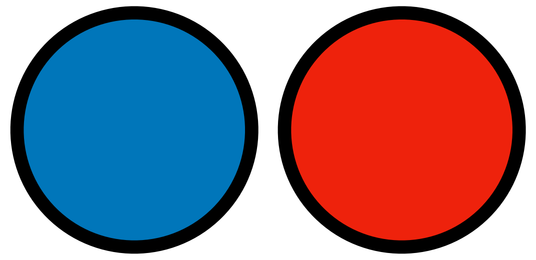

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  - If it does not enter the accept or reject states, TM will run forever (***loop***), and never halt

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- We can use Turing machines to solve problems!

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  - Does 1100010000000010 contains 2^k 0s for some integer k?

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1. Sweep left to right across the tape, crossing off every other 0.

2. If in stage 1 the tape contained a single 0, *accept*.

3. If in stage 1 the tape contained more than a single 0 and the number of 0s was odd, *reject*.

4. Return the head to the left-hand end of the tape.

5. Go to stage 1."

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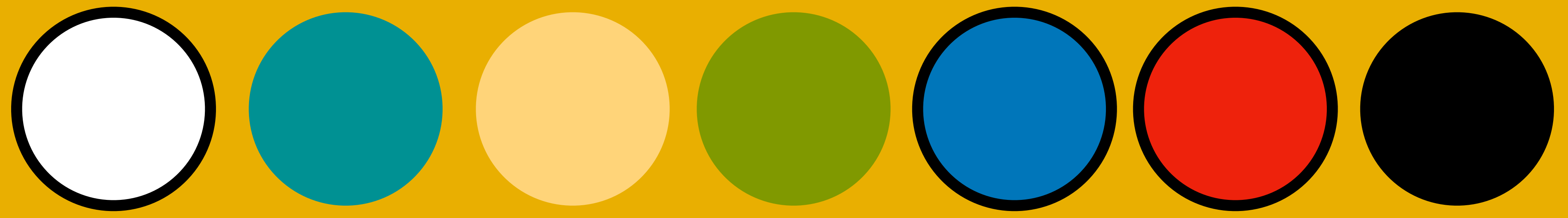
- Finite-state control 
  - The Turing machine may end up with an *accept* state or *reject* state
  - It *accepts* the input or *rejects* the input

Nondeterministic Turing Machine

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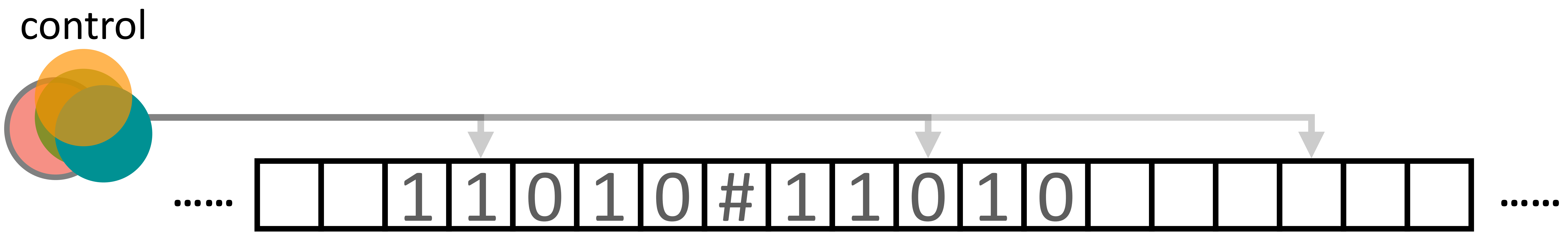

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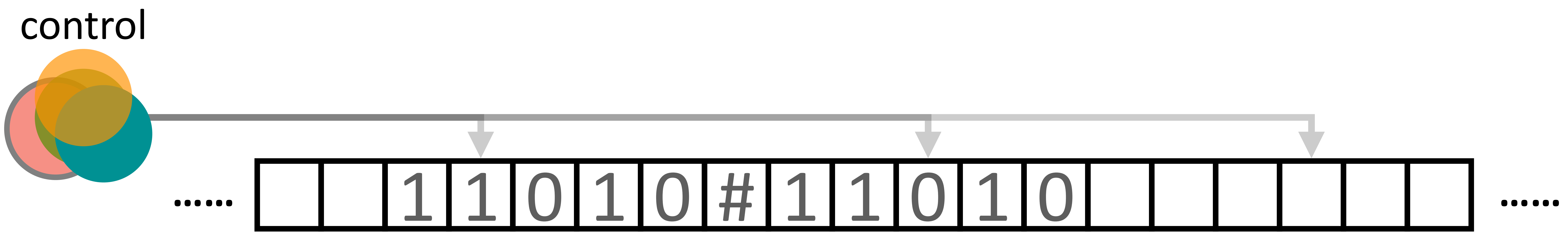

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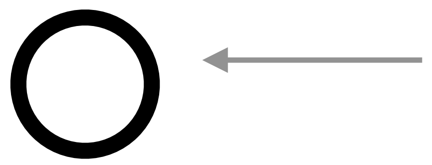
- For an input w, we can describe all possible computations of nondeterministic Turing machine by a *computation tree*

Nondeterministic Turing Machine

Deterministic



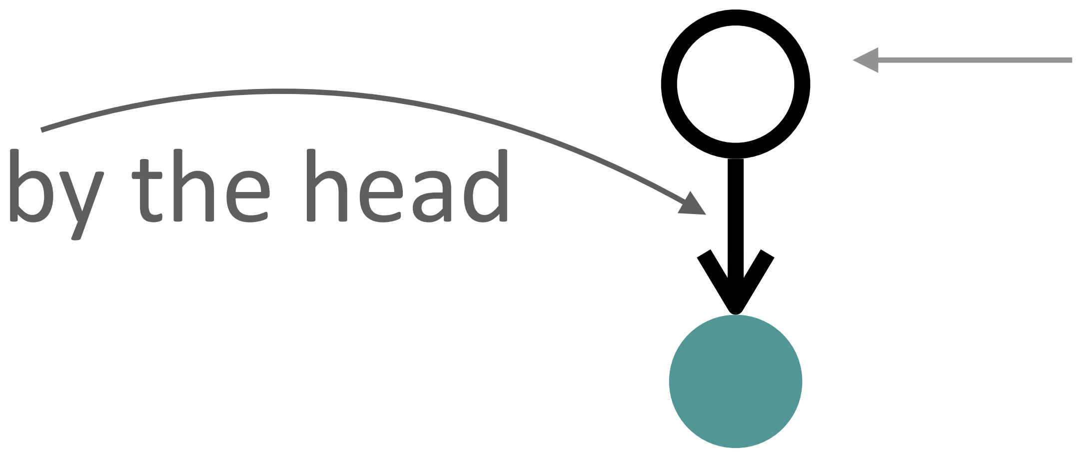
Root: initial configuration

: Configuration

(The current control state and
what the read-write head reads

Nondeterministic Turing Machine

Deterministic



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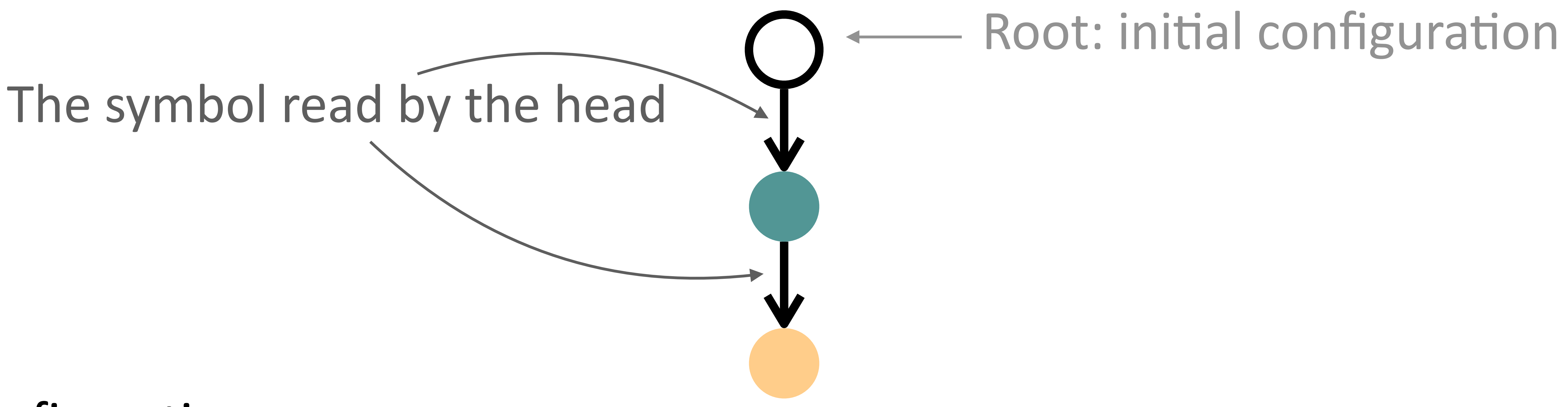

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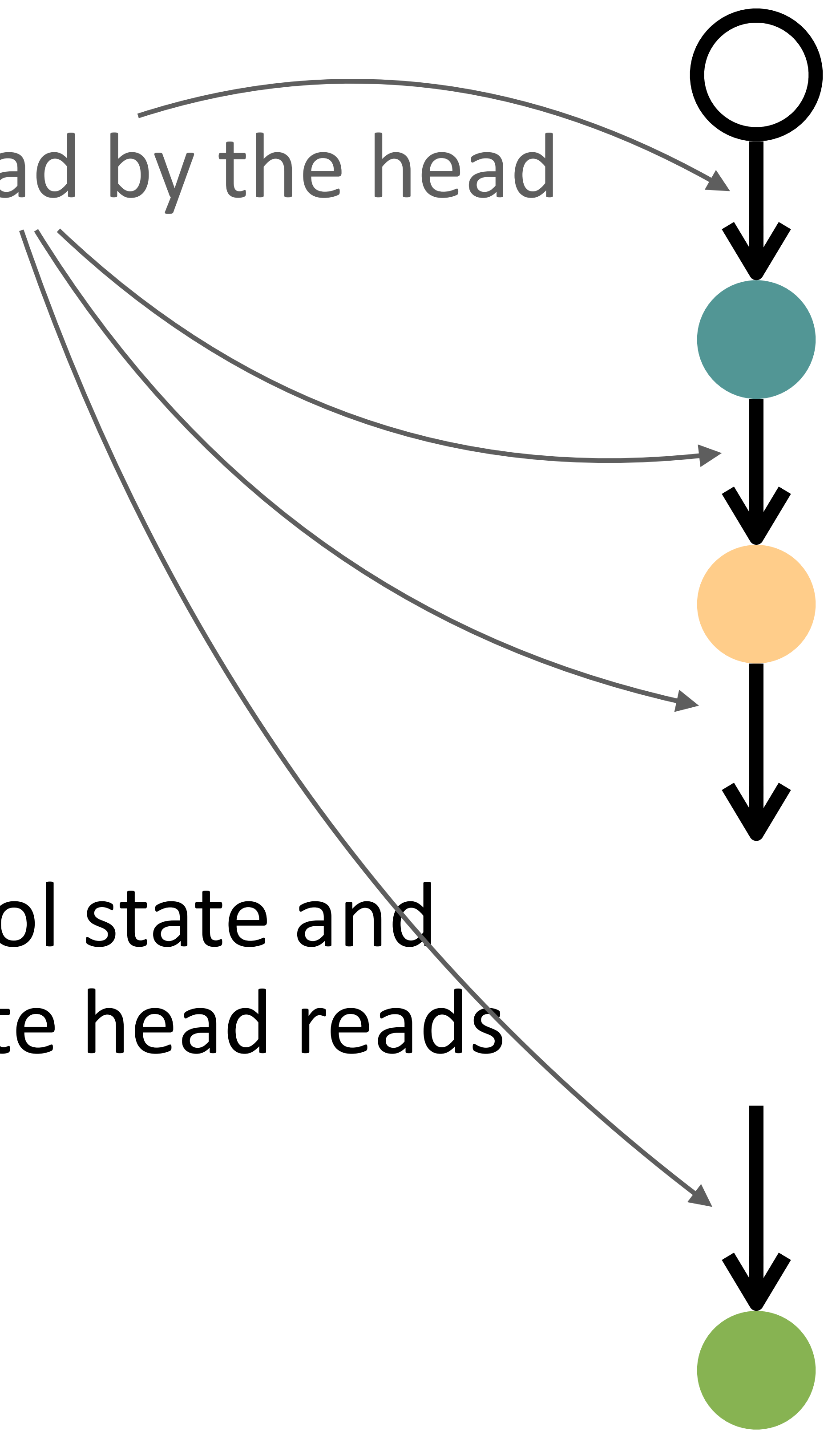

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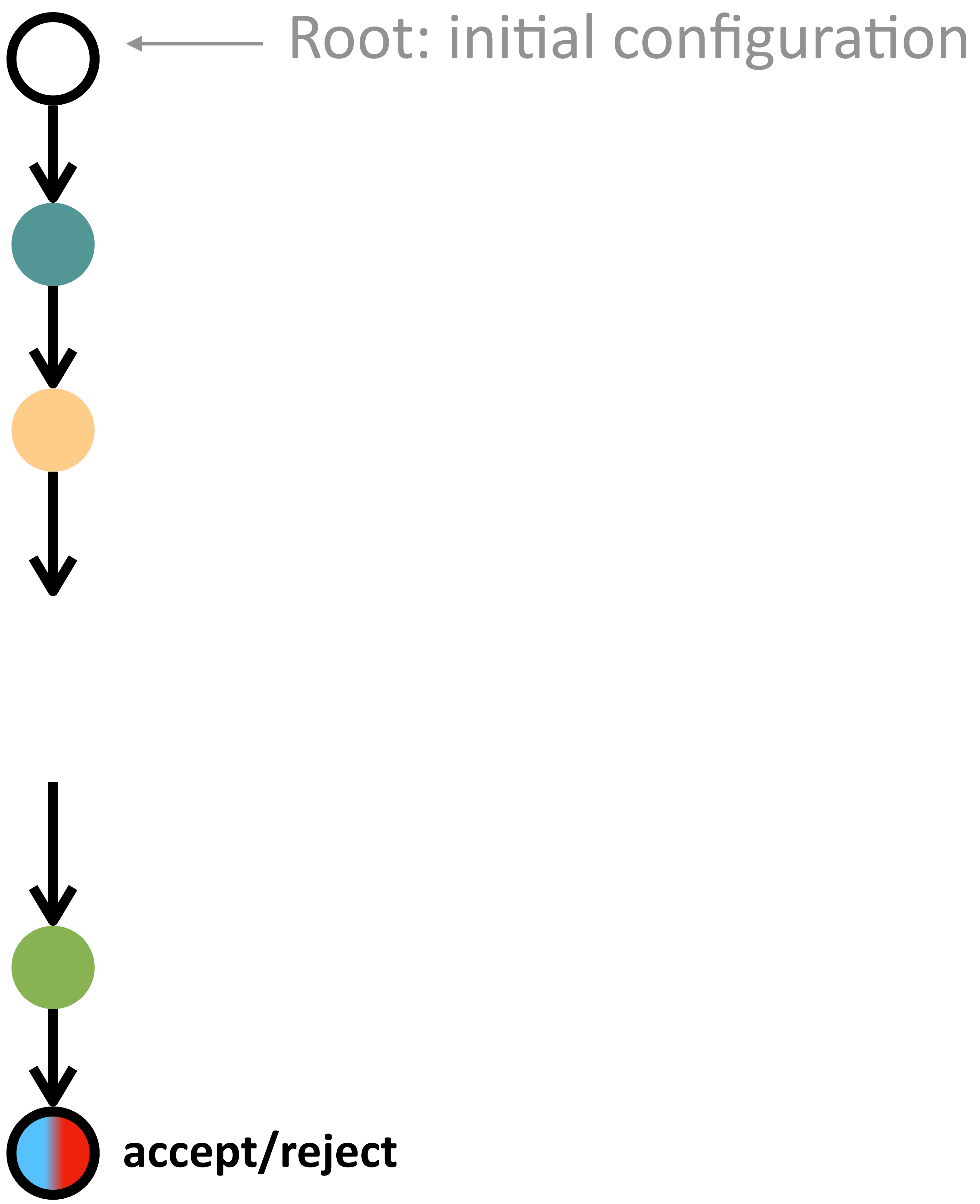

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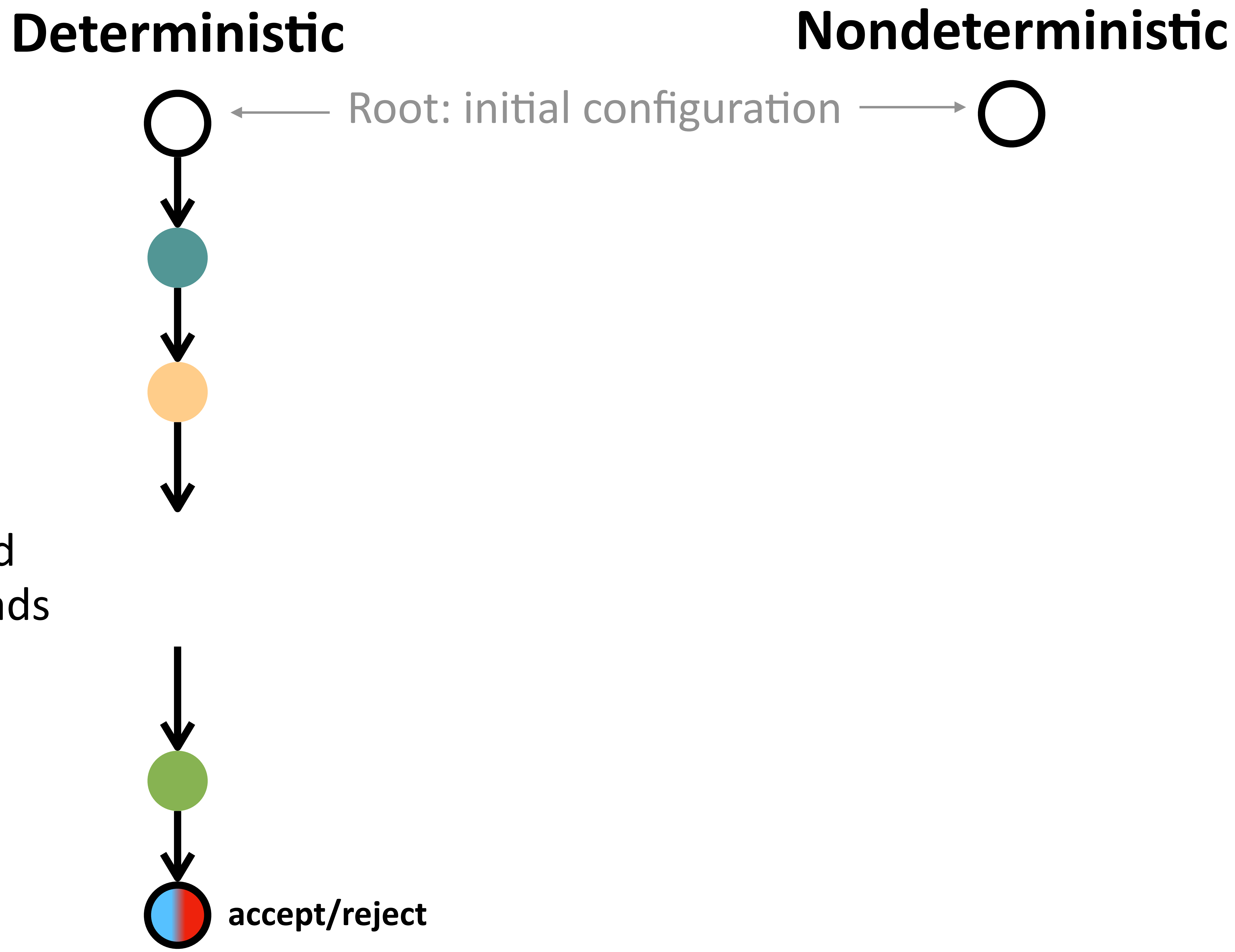

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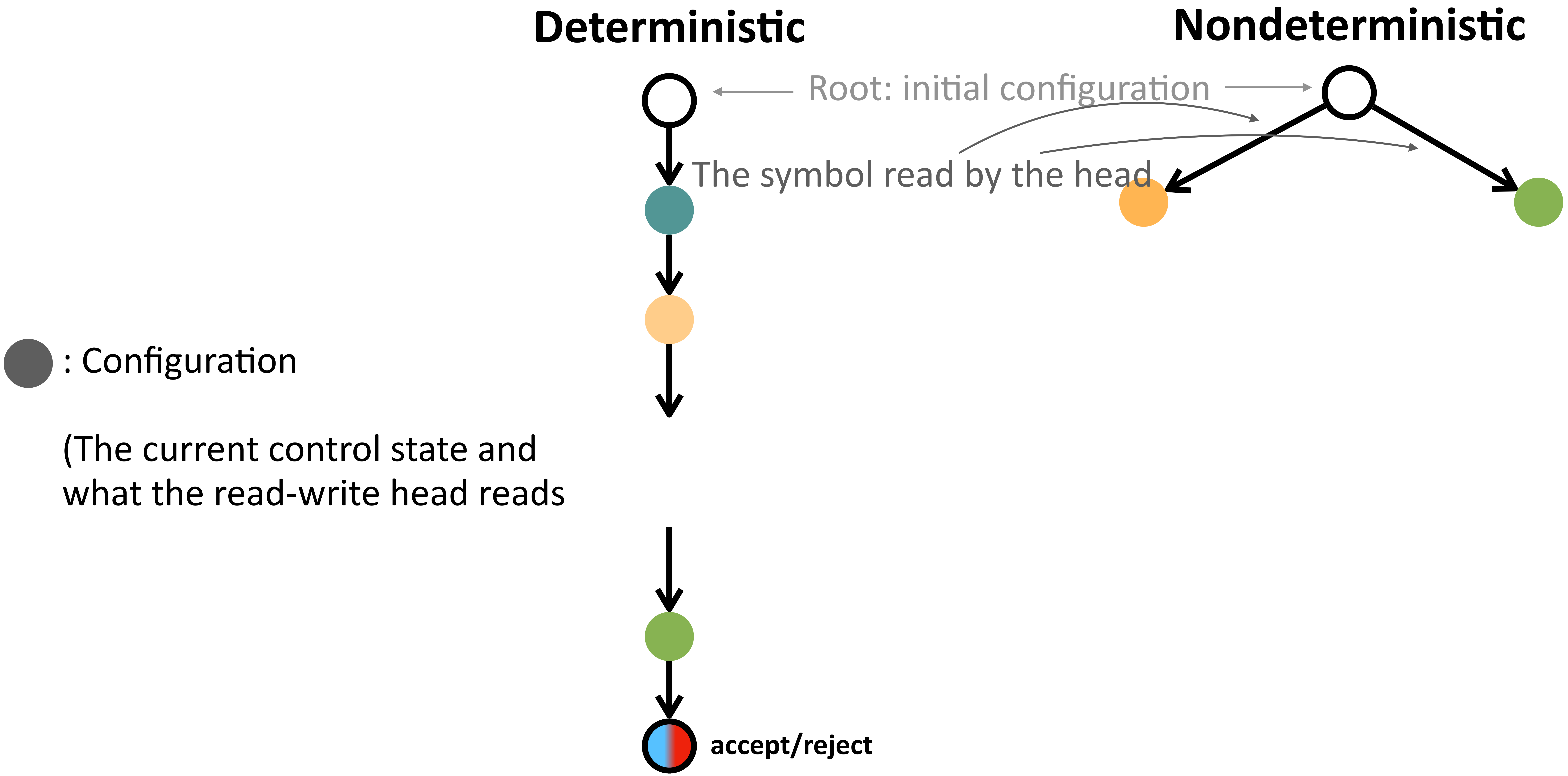

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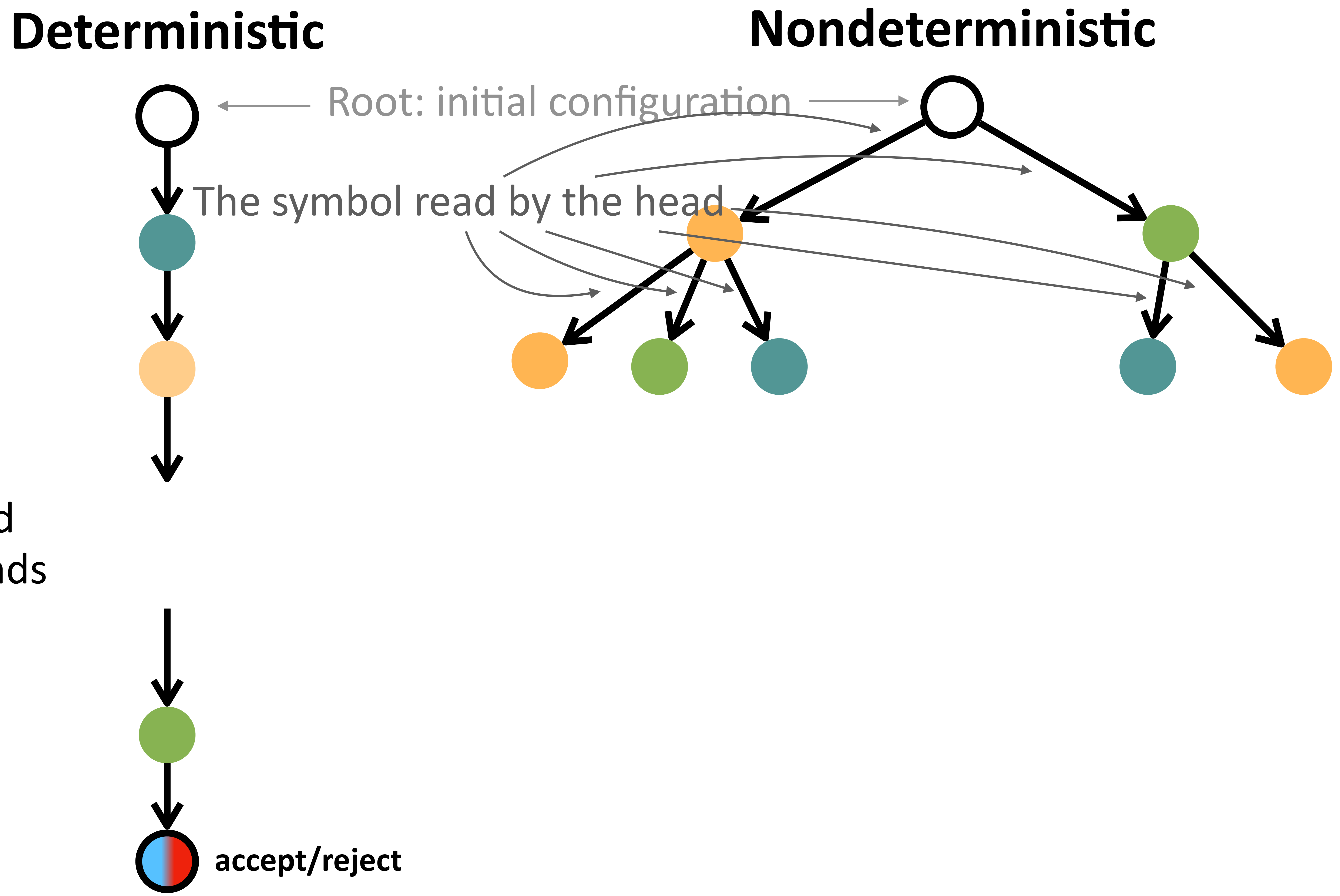

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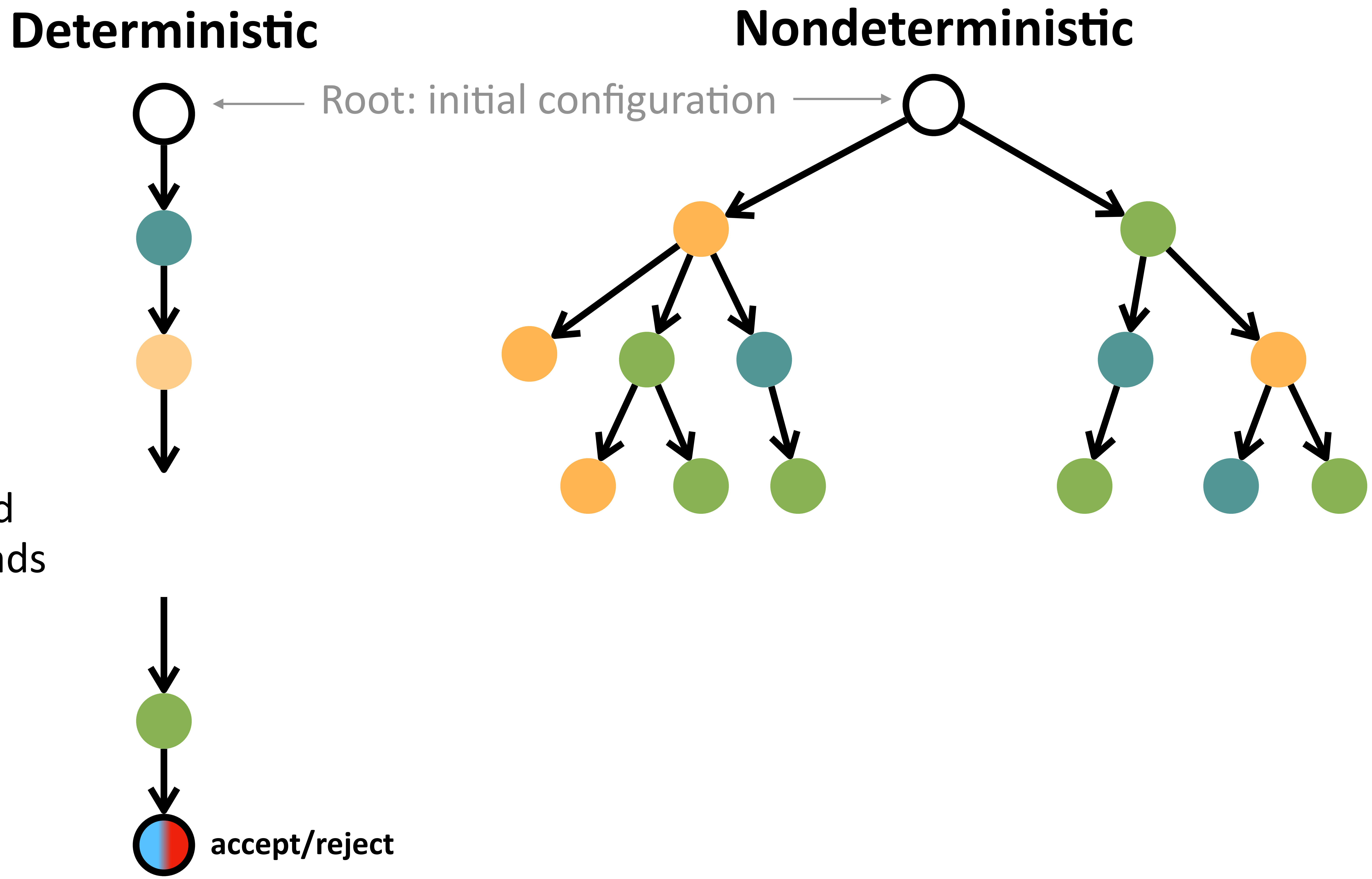

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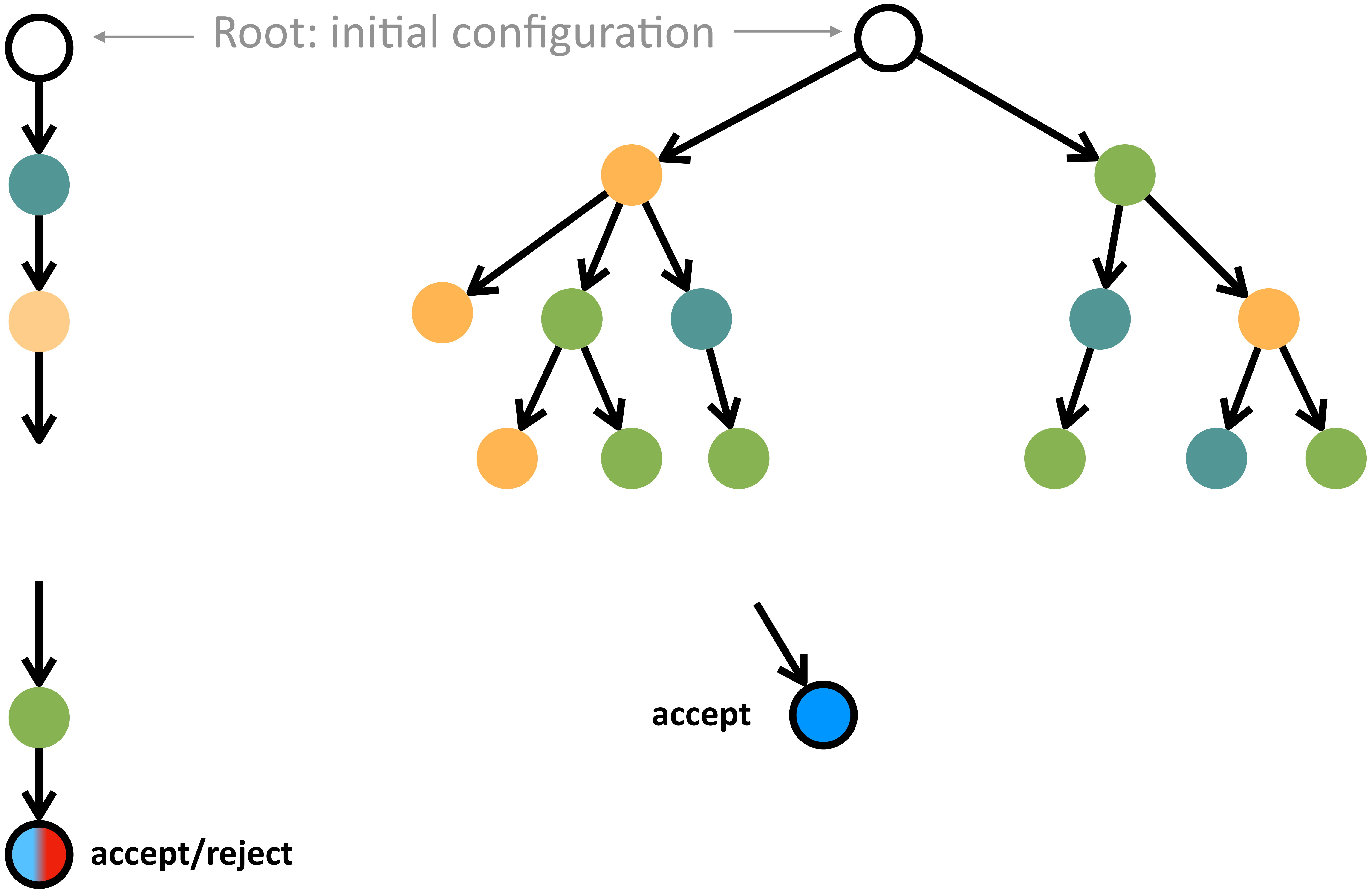
Deterministic **Nondeterministic**



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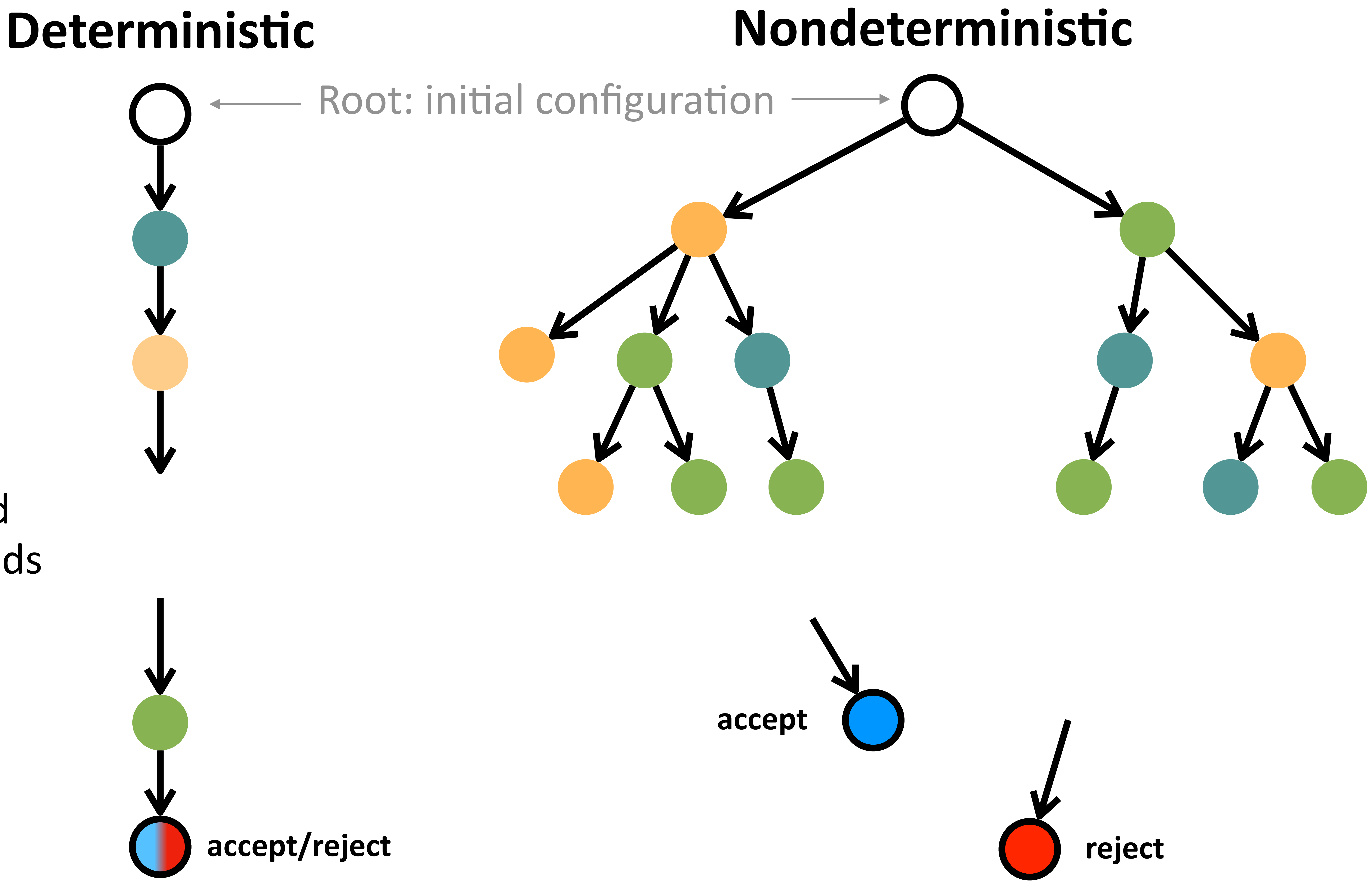

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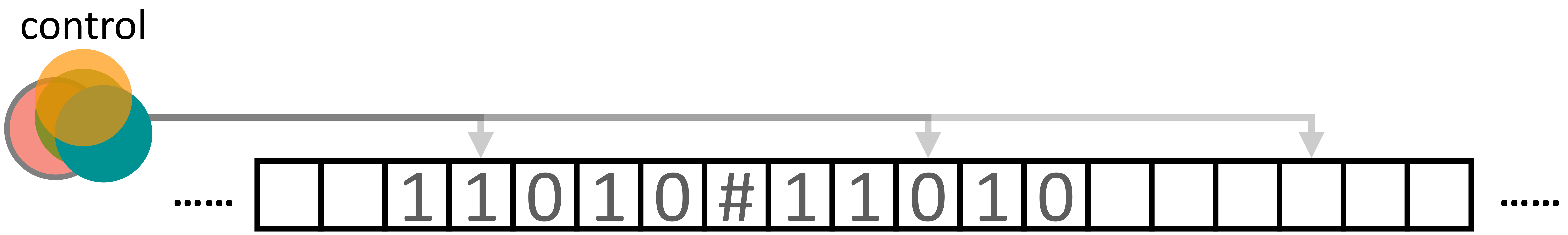

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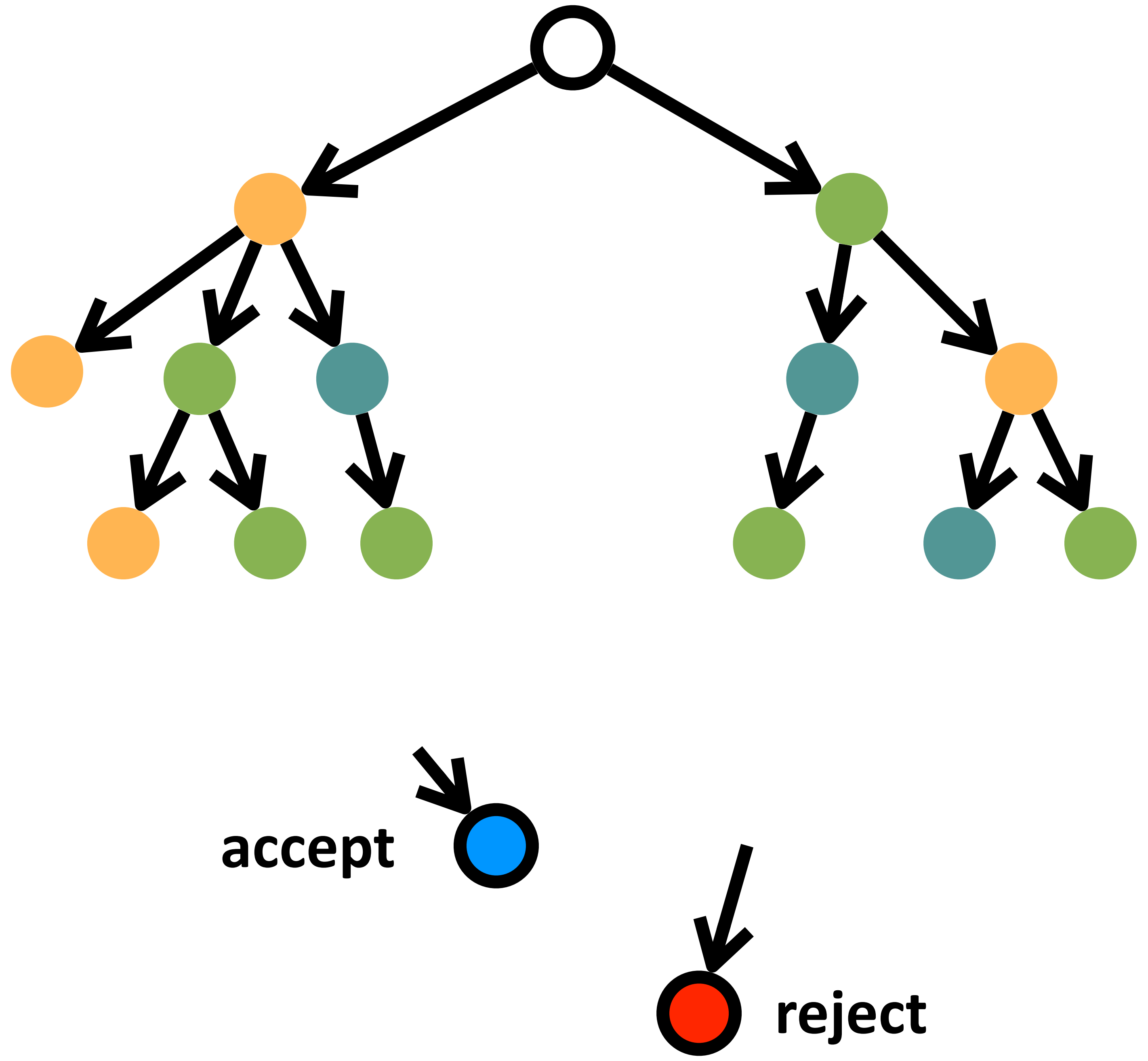
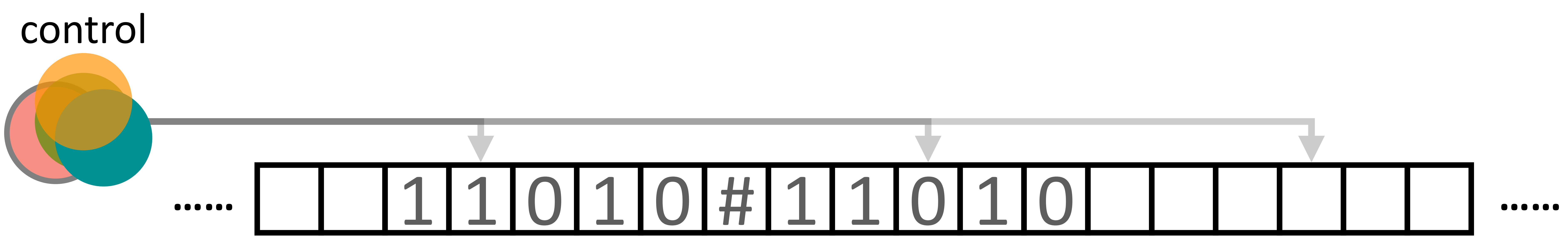


Nondeterministic Turing Machine



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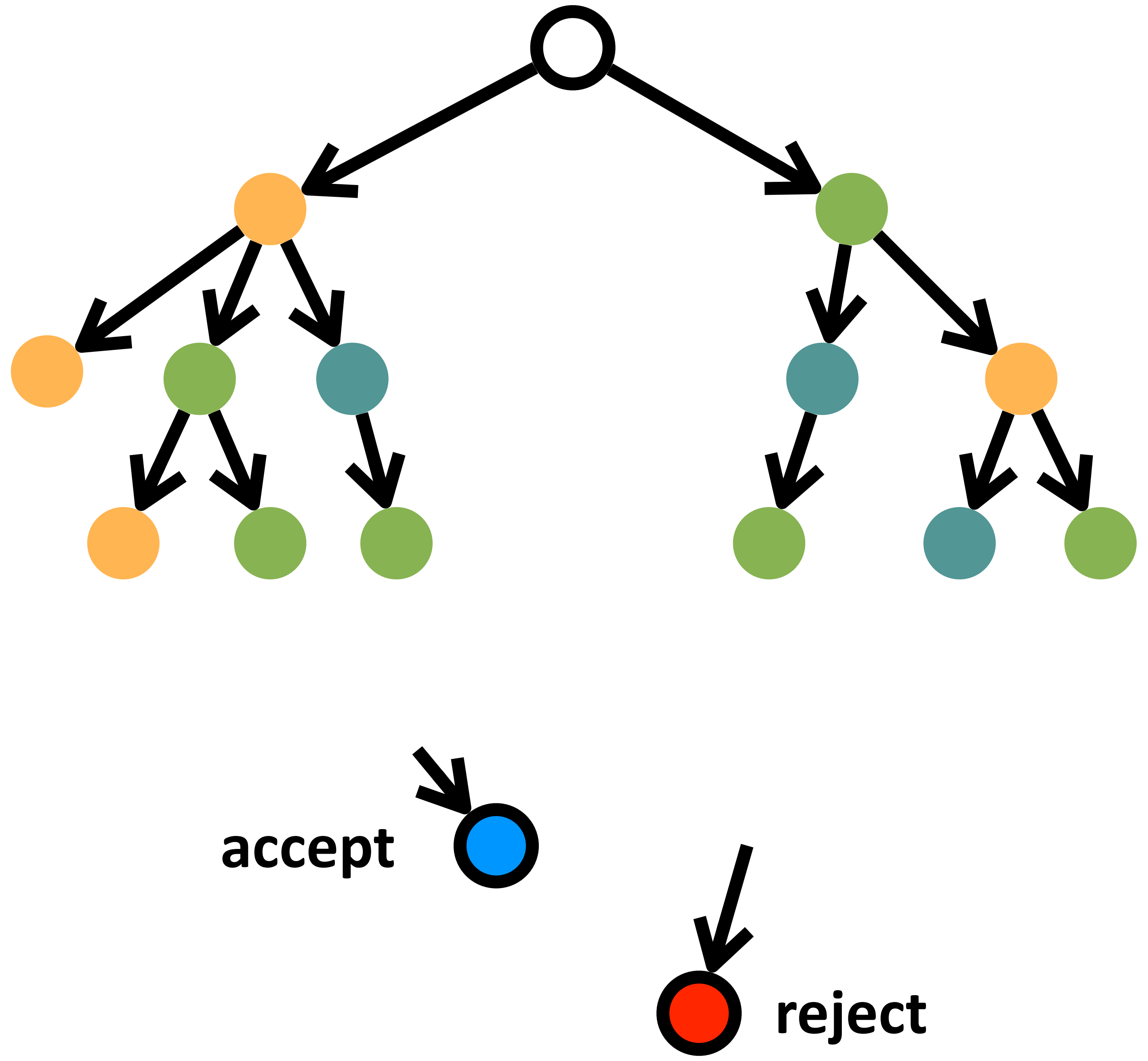
  - Given an state and read a symbol, the non-deterministic Turing machine may enter different states



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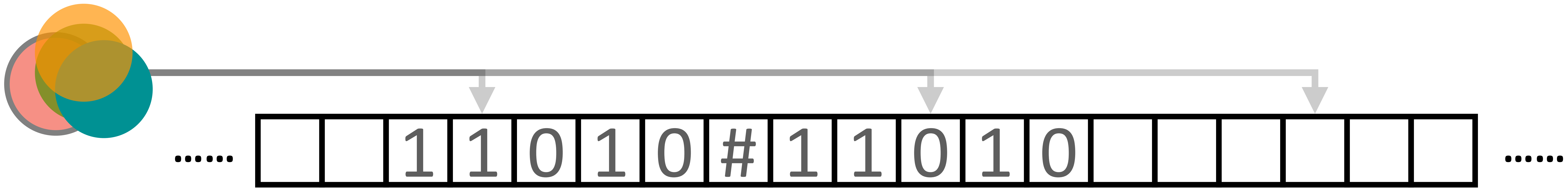
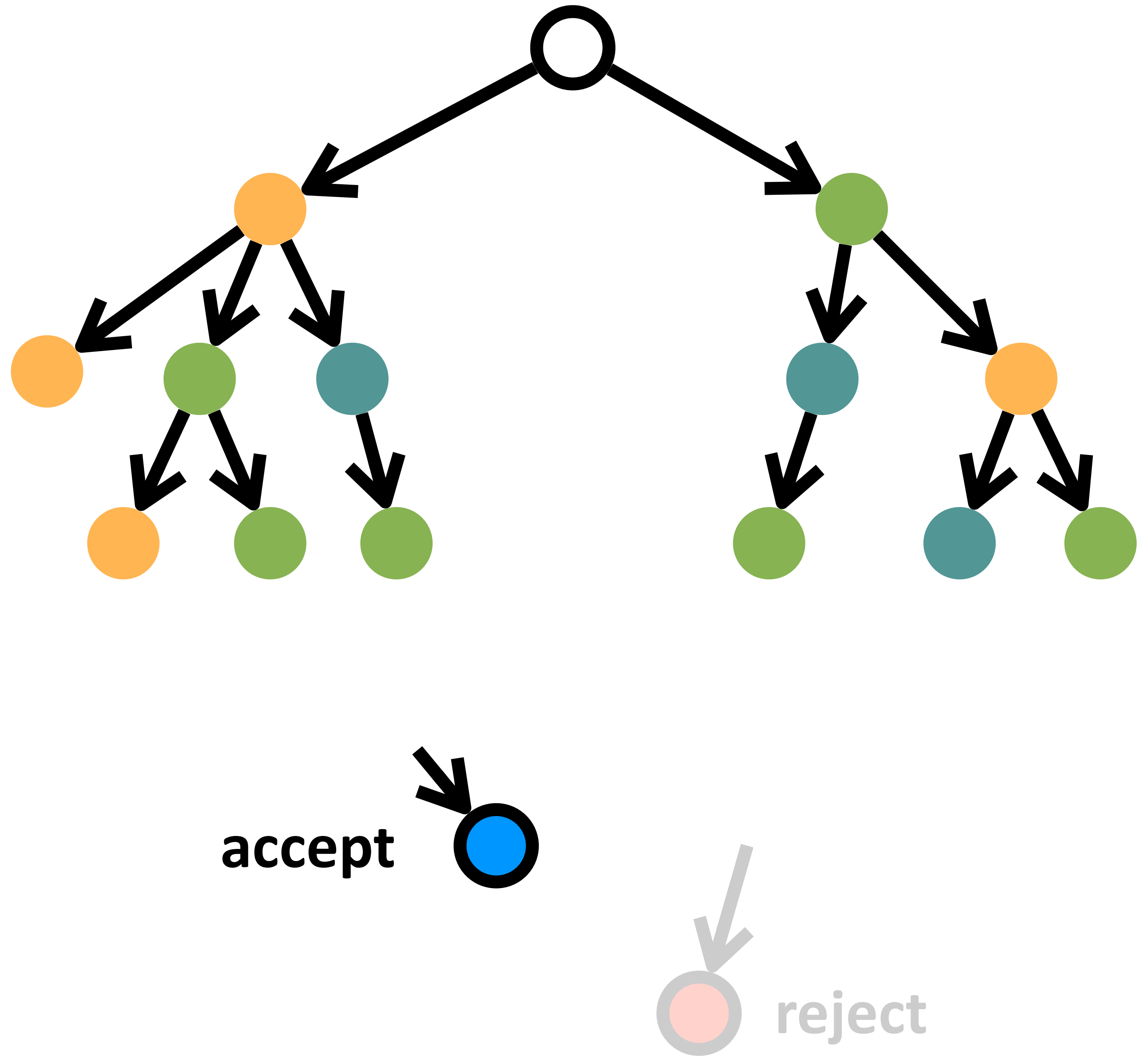
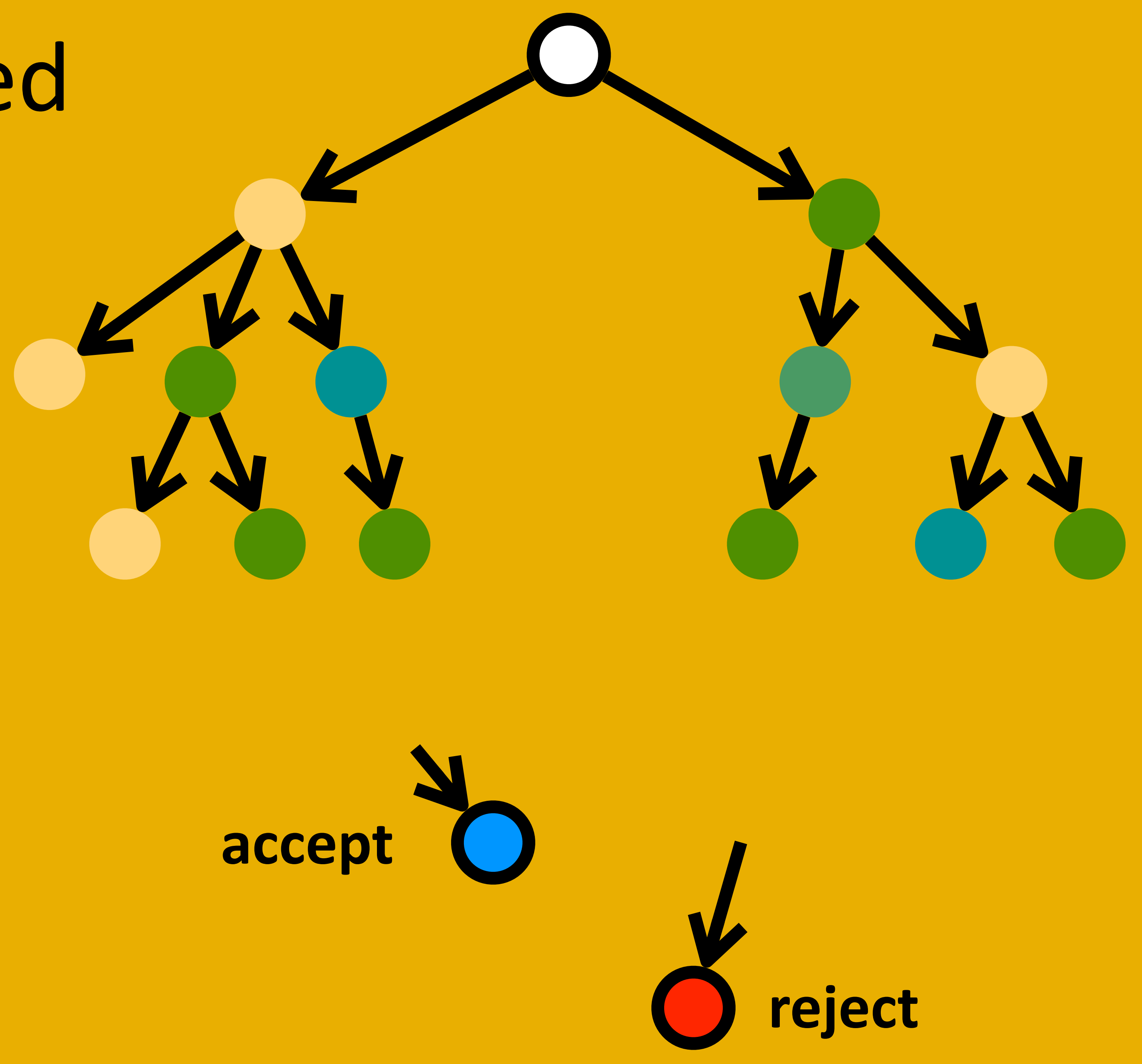


- It is like a (deterministic) Turing machine, but with non-deterministic control
  - Given an state and read a symbol, the non-deterministic Turing machine may enter different states
- The nondeterministic Turing machine ***accepts*** the input w if **some** branch of computation (i.e., a path from root to some node) leads to the **accept** state



Non-Deterministic Turing machine

- Like the (deterministic) Turing machine, but have non-deterministic behavior
- If there is **a** path ends at an accept state, the input is accepted



What is an algorithm/computer?

Church-Turing Thesis [1936]

Real world computation = Turing machine computation

Overview

- Algorithms:Turing machine, Deterministic and non-deterministic
- Formal language framework: string and language
- Time complexity
  - Input size
  - Classes **P** and **NP**
    - Polynomial time verification

A Formal Language Framework for *Problems*

- Initially, there is a string of symbols on the Turing machine tape

		1	1	0	1	0	#	1	1	0	1	0						

A Formal Language Framework for *Problems*

- Definition: A ***language*** is a set of strings (that satisfy some constraints)
  - $\{1, 01, 001, 0001, 00001, \ldots, 0^{*}1\}$
  - $\{0, 1\}^*$ Example: $010, 0, 1, 11111, \phi, \cdots$

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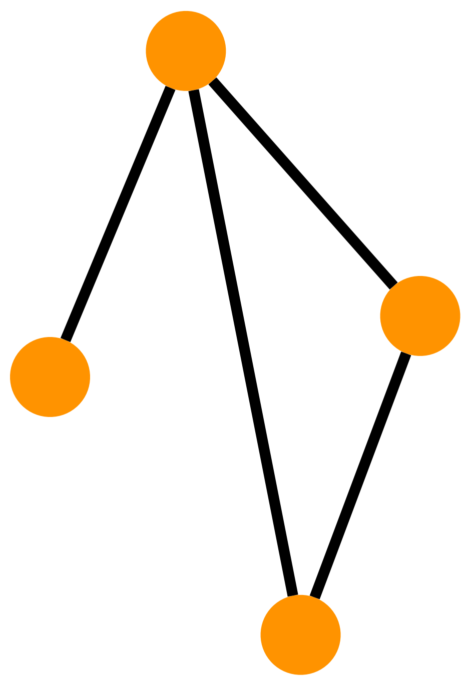

We use $\langle \cdot \rangle$ to represent an encoding of $\cdot$

(binary!)

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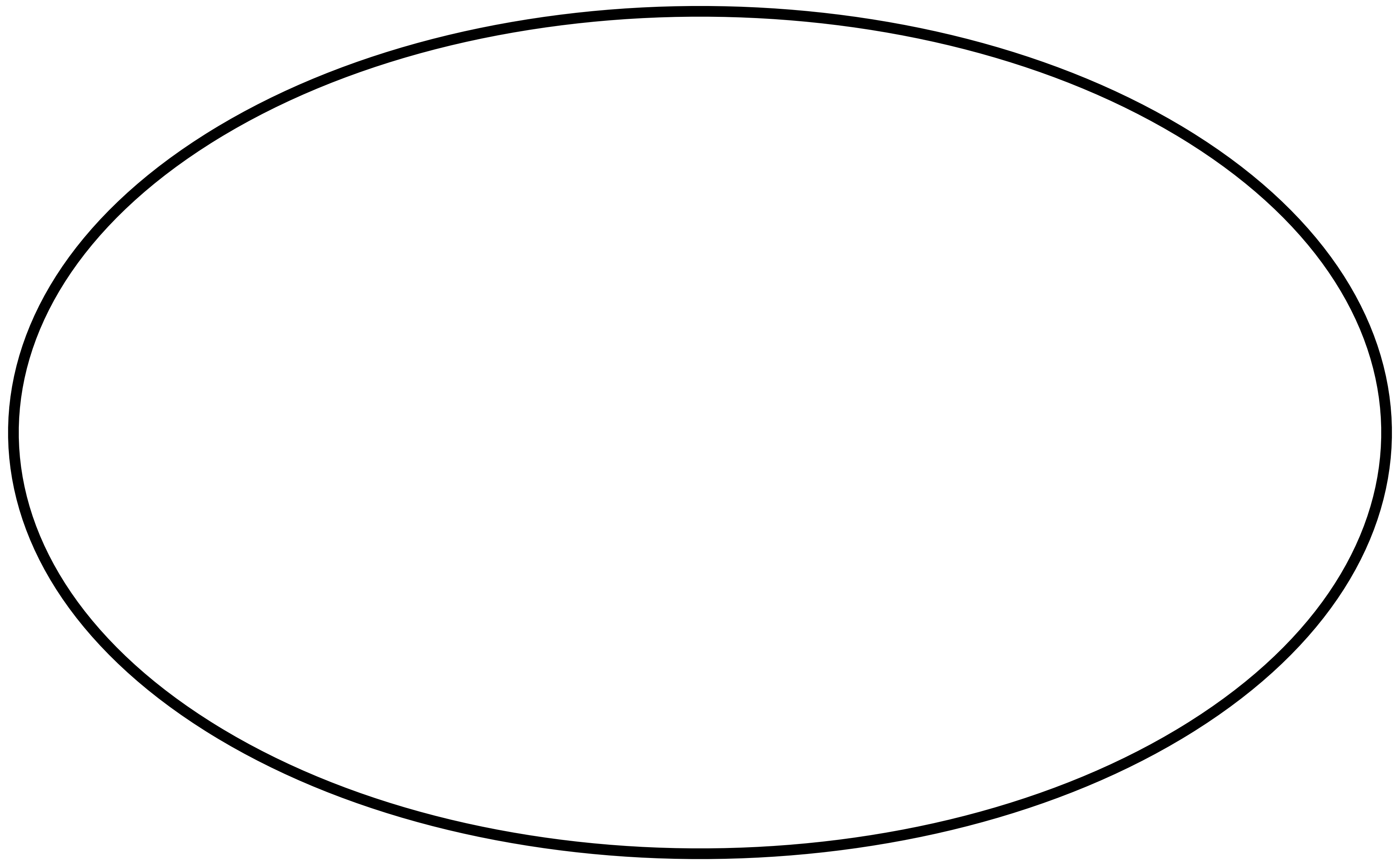

all possible strings

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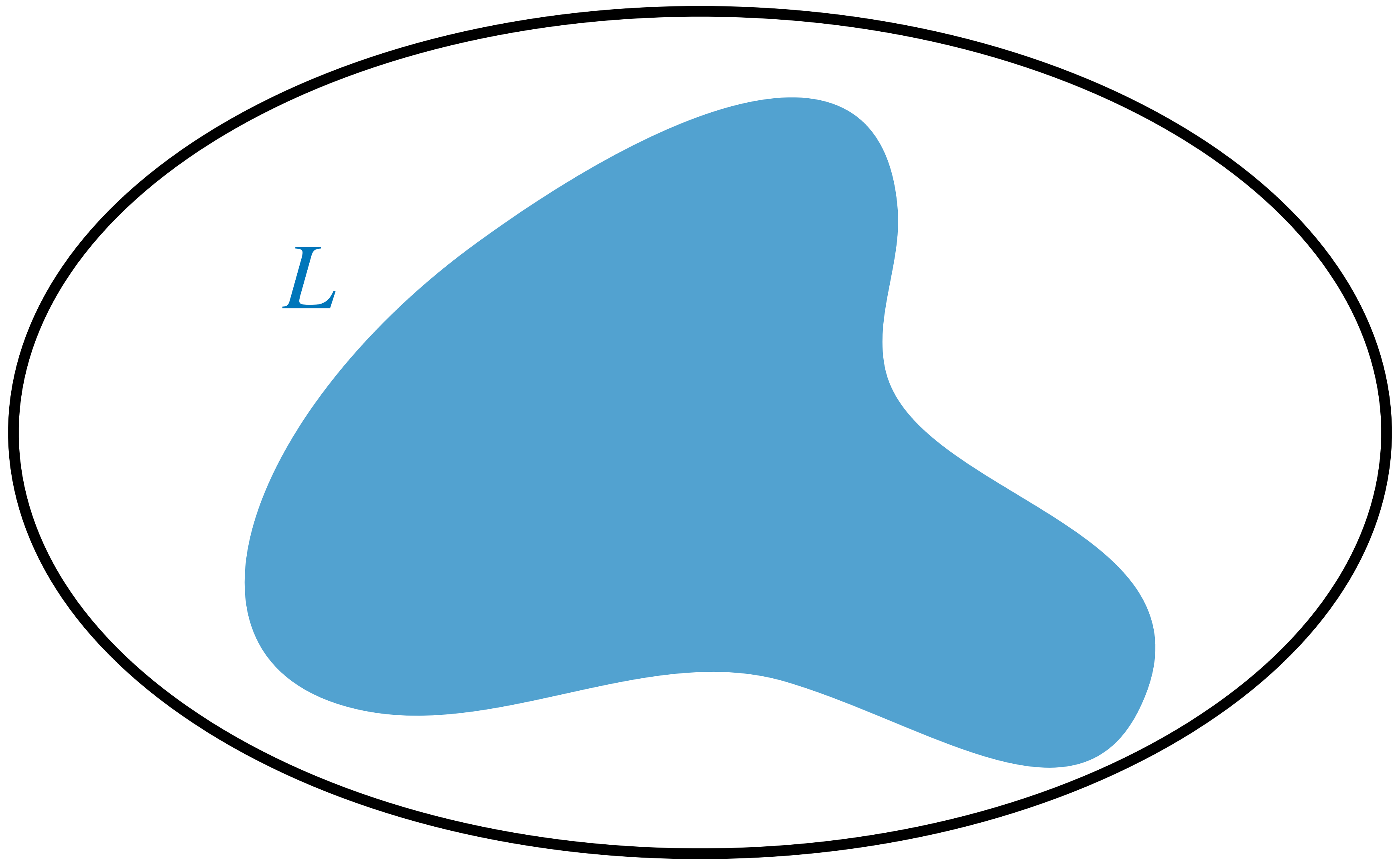

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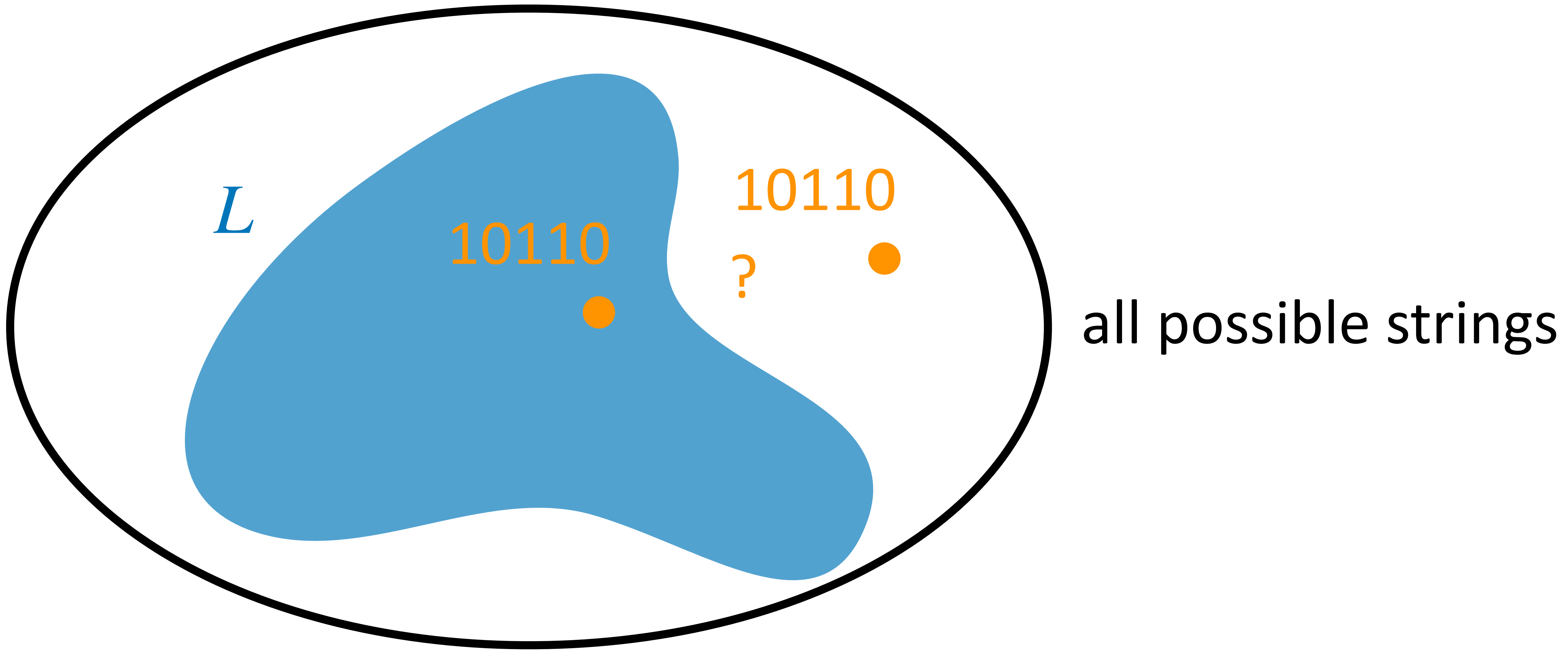

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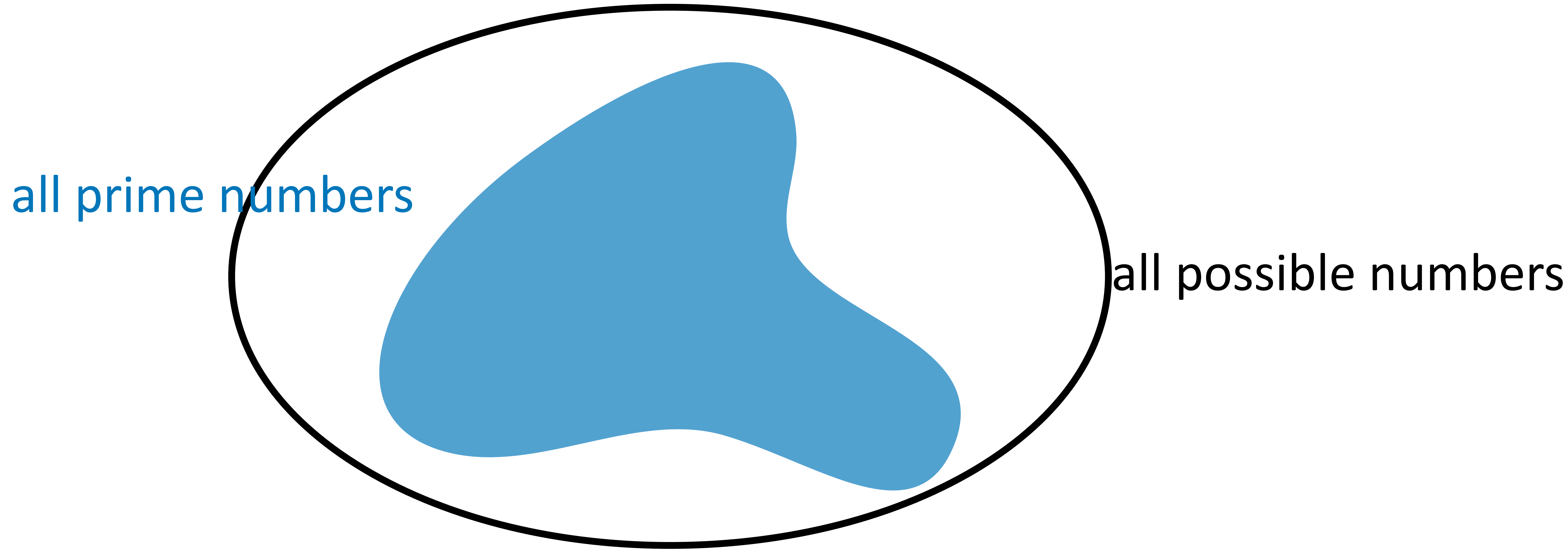

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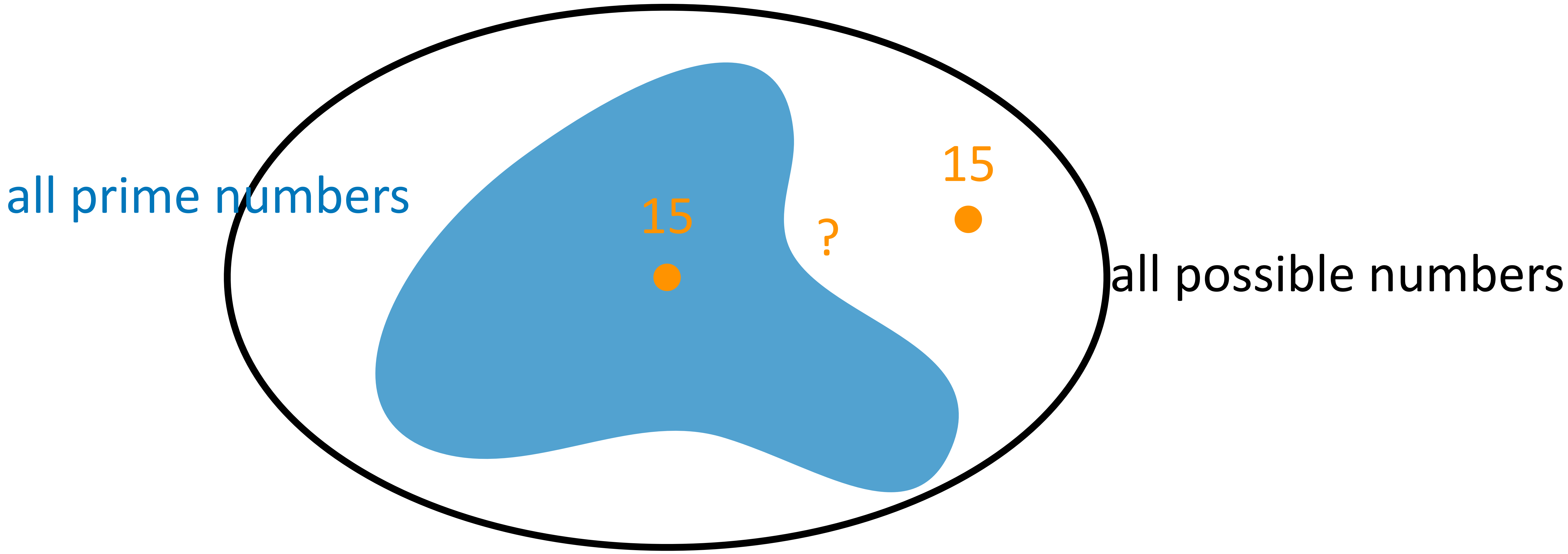

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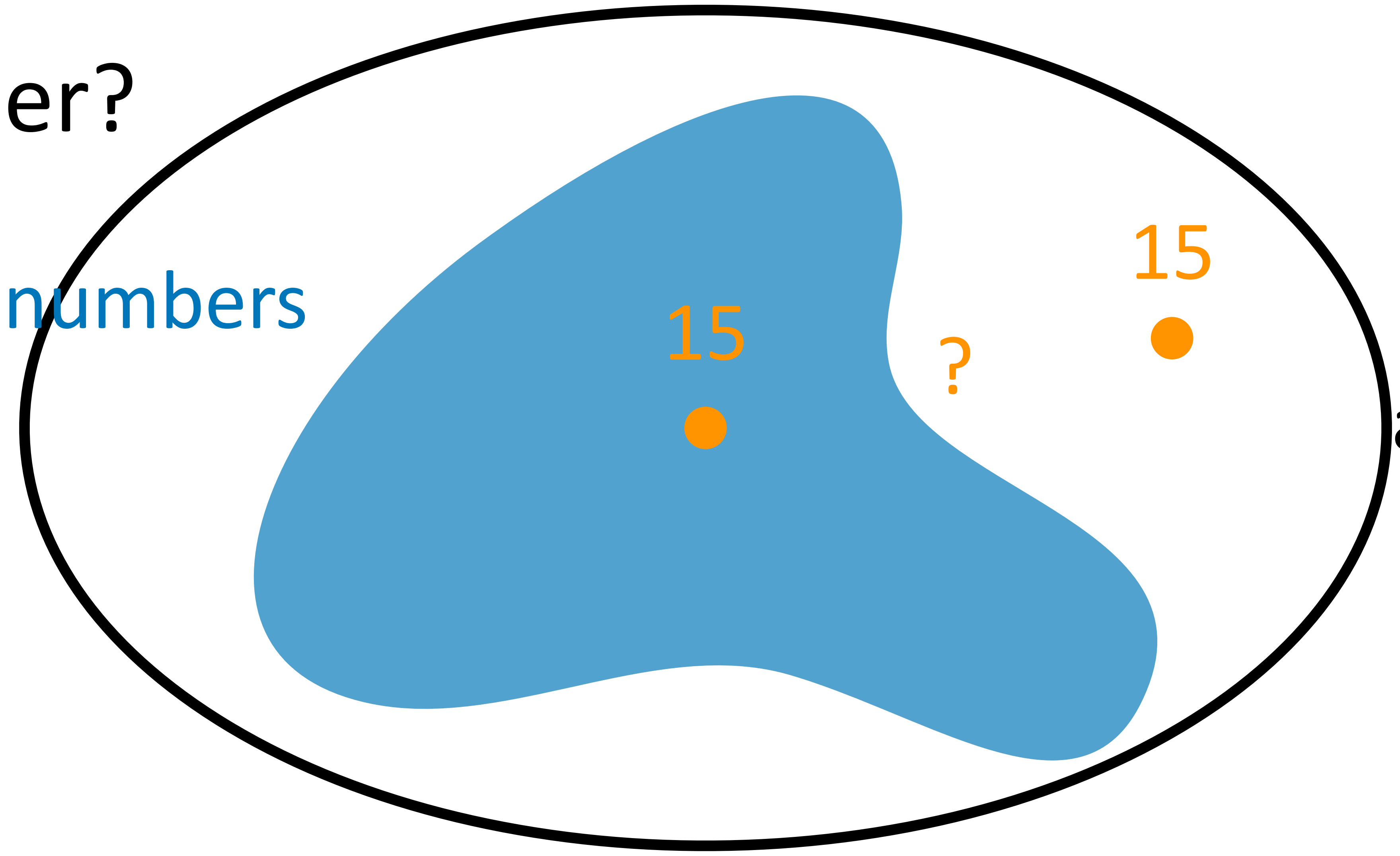

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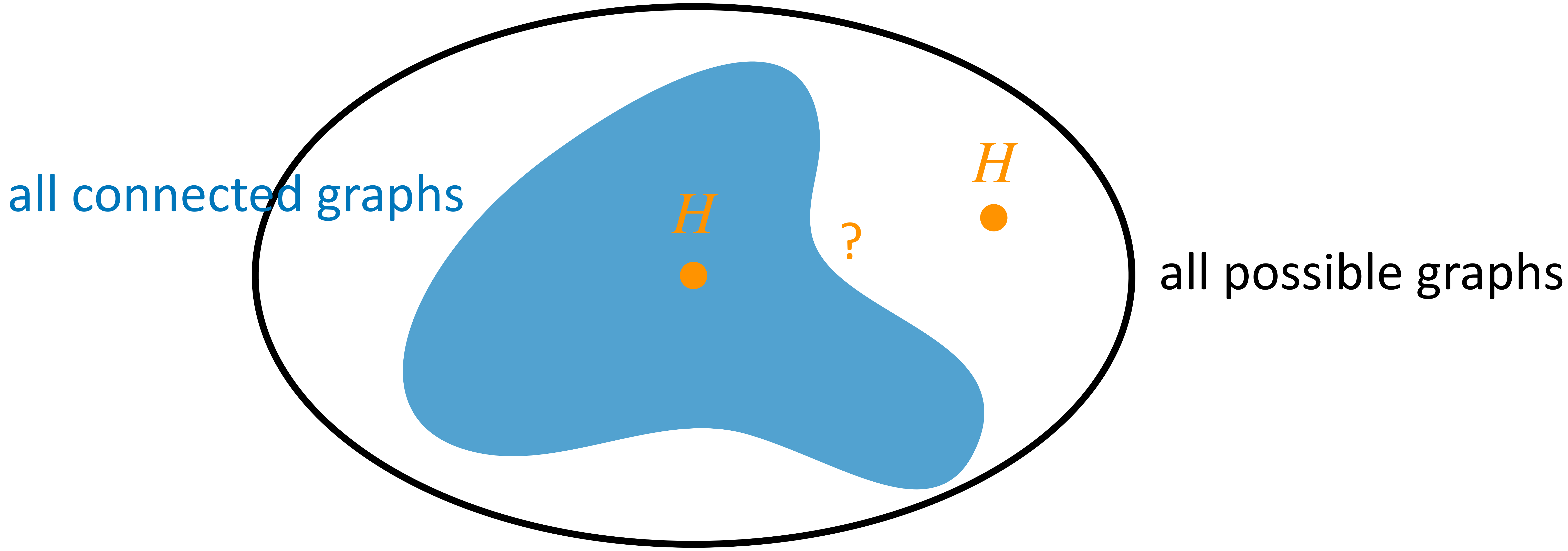

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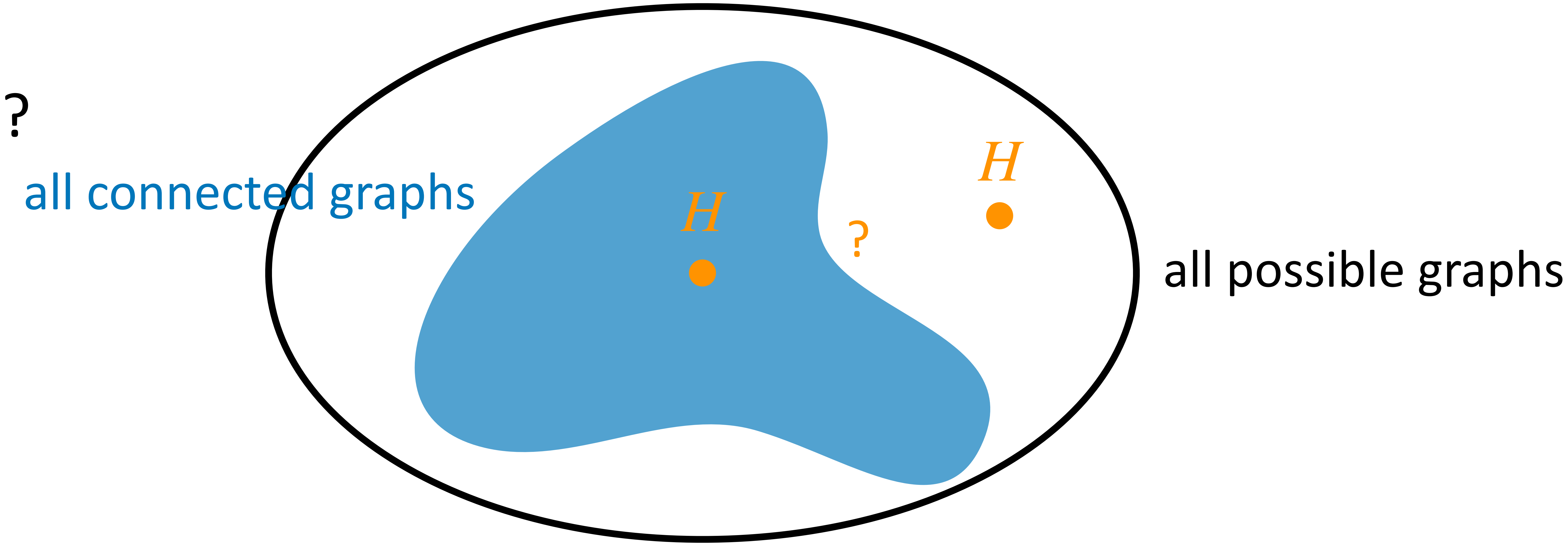

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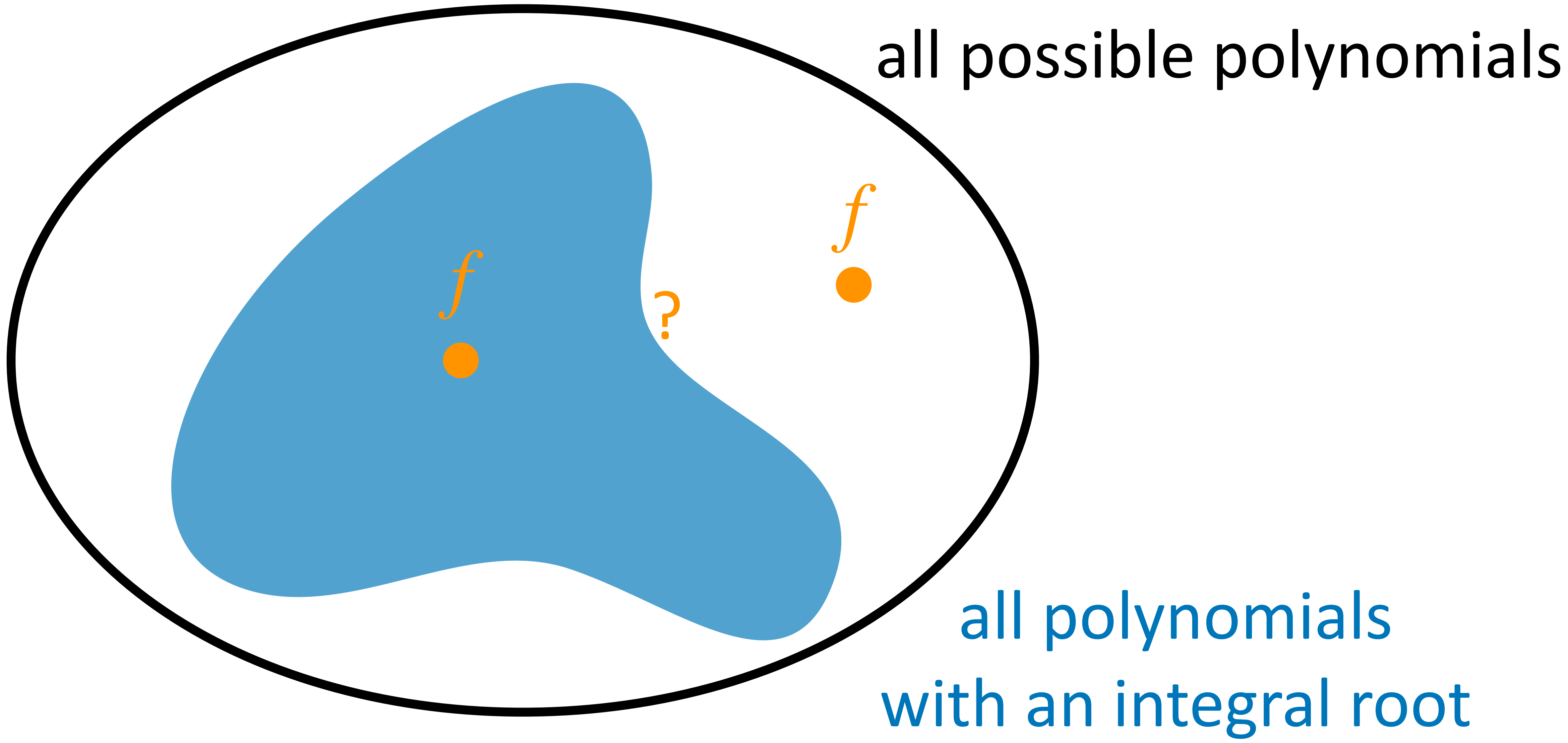

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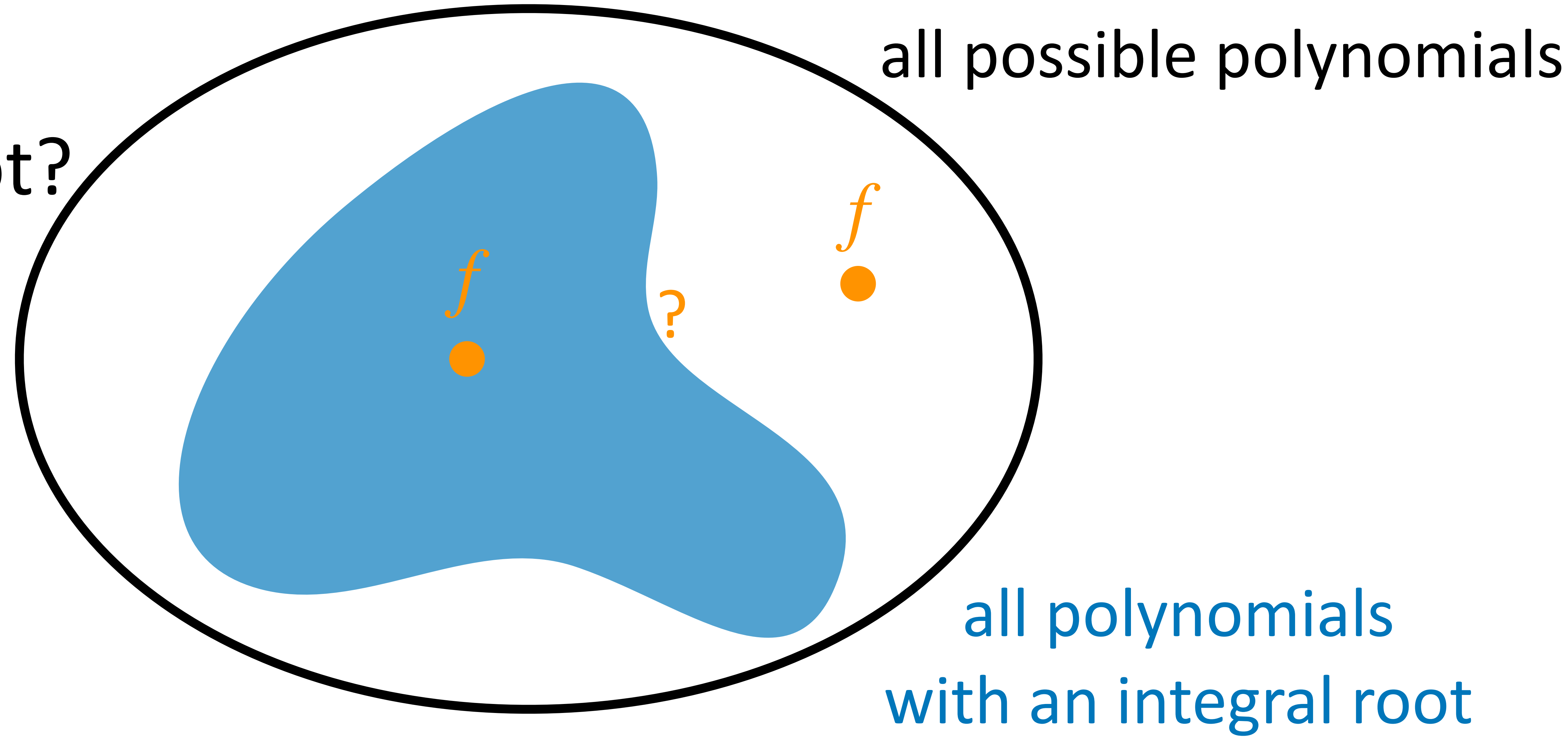

Yes-Instance and No-Instance

- Consider a language A. For any instance w, either $w \in A$ or $w \notin A$.
  - If $w \in A$, we call w a **yes-instance** (that is, a correct algorithm should return *accept* or *yes*)
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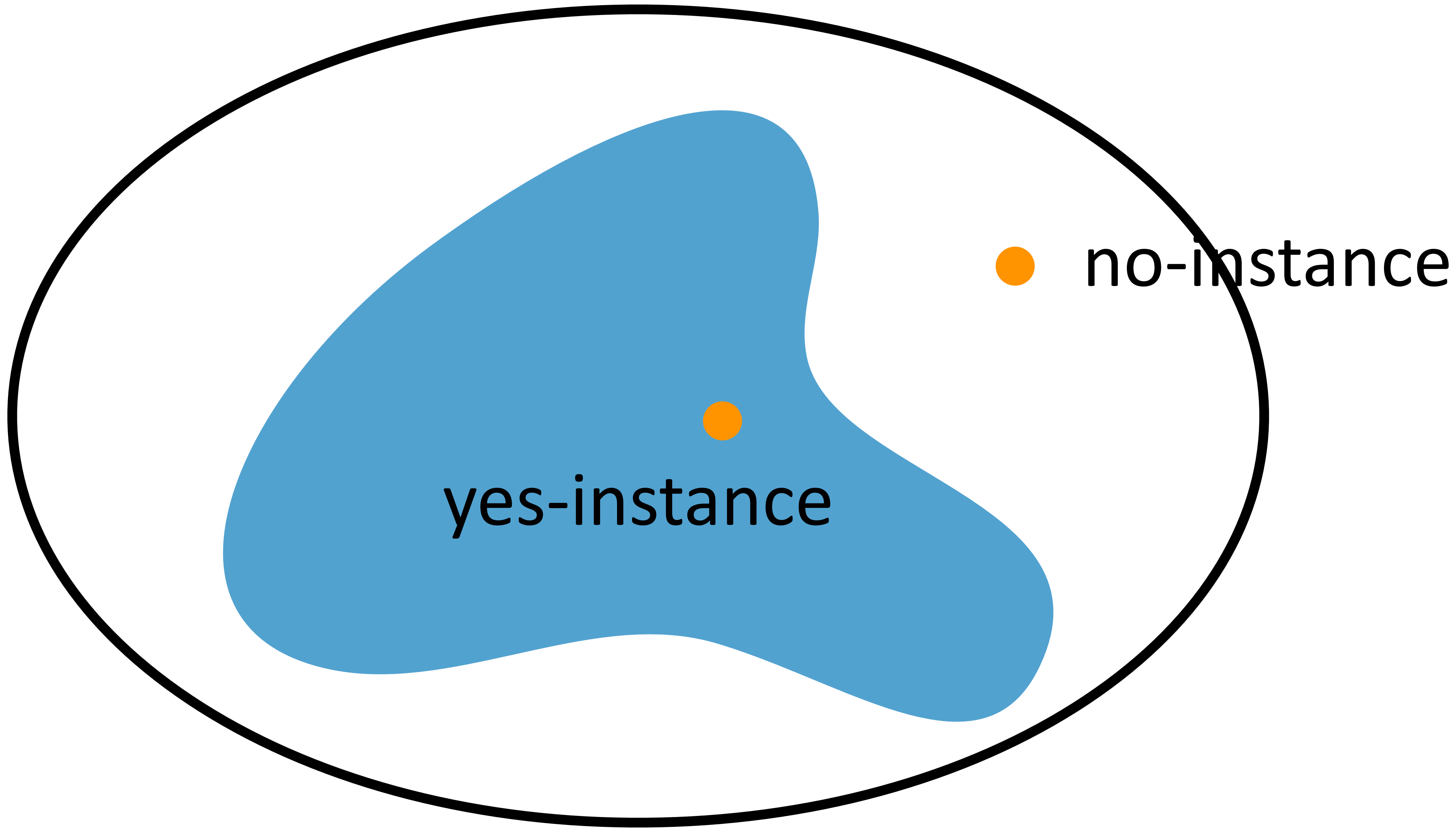

What Happened

- Following the vein of Turing machine concept, a **language** is a set of **strings**
  - Language ⇔ **problem**
  - String ⇔ **instance**
  - Asking if a string is in a language
    ⇔ if the instance satisfies the property that the problem asks

- Given a problem/language, a instance/string is a
  - **yes-instance**: an instance that satisfies the property that the problem asks
  - **no-instance**: an instance that does not satisfy the property that the problem asks

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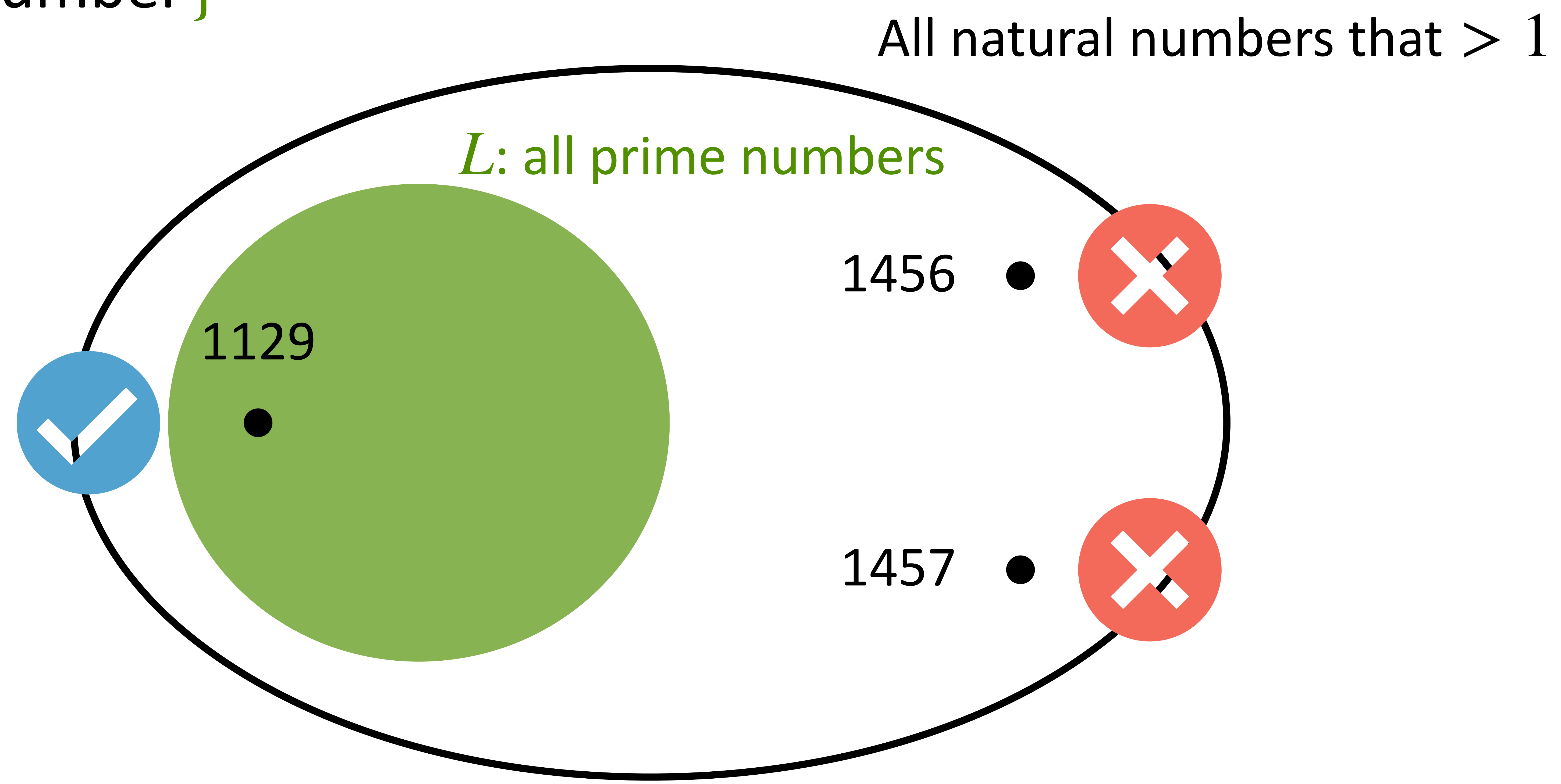
- Ex: L = $\{$prime number$\}$



Decision Problems and Optimization Problems

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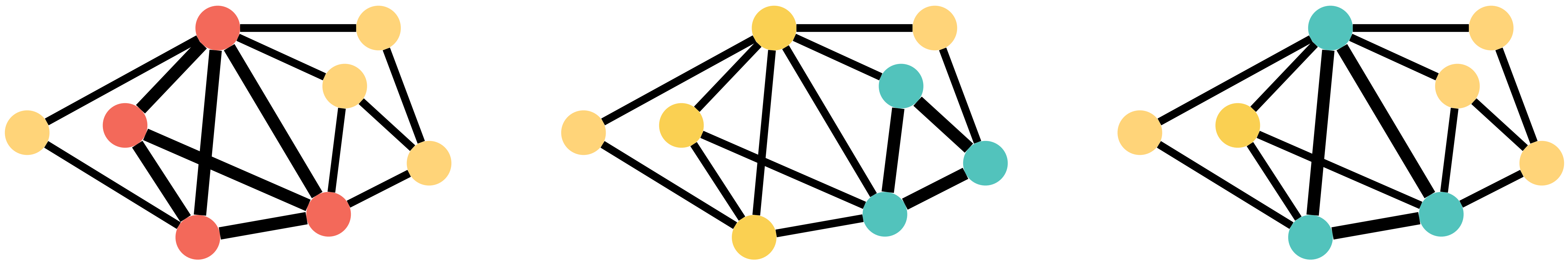

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      - Ex: Minimum vertex cover or Maximum independent set

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    - k is an additional parameter

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- Algorithms:Turing machine, Deterministic and non-deterministic

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- Time complexity
  - Input size
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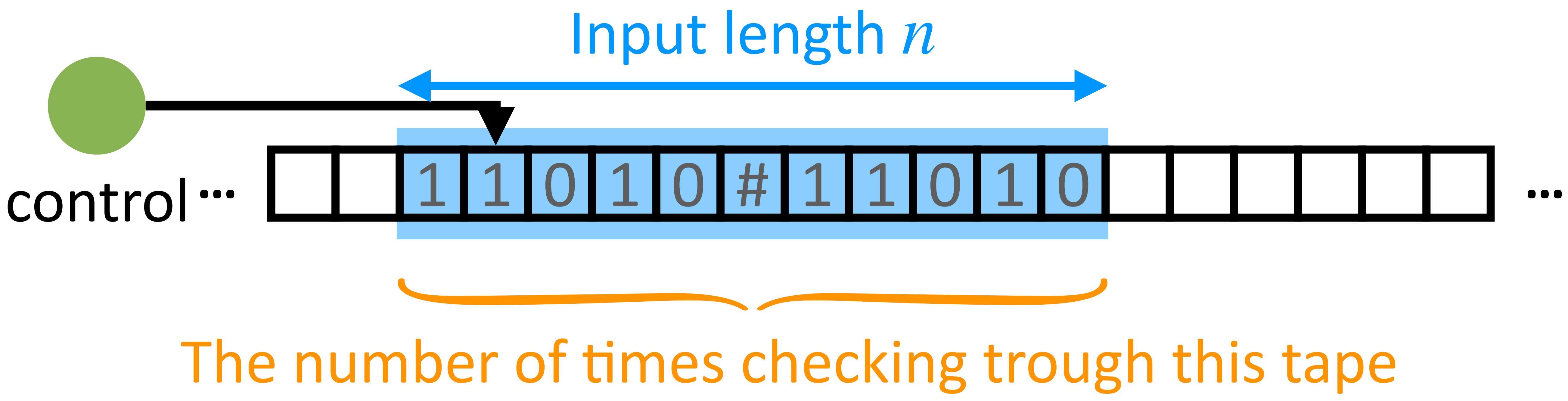
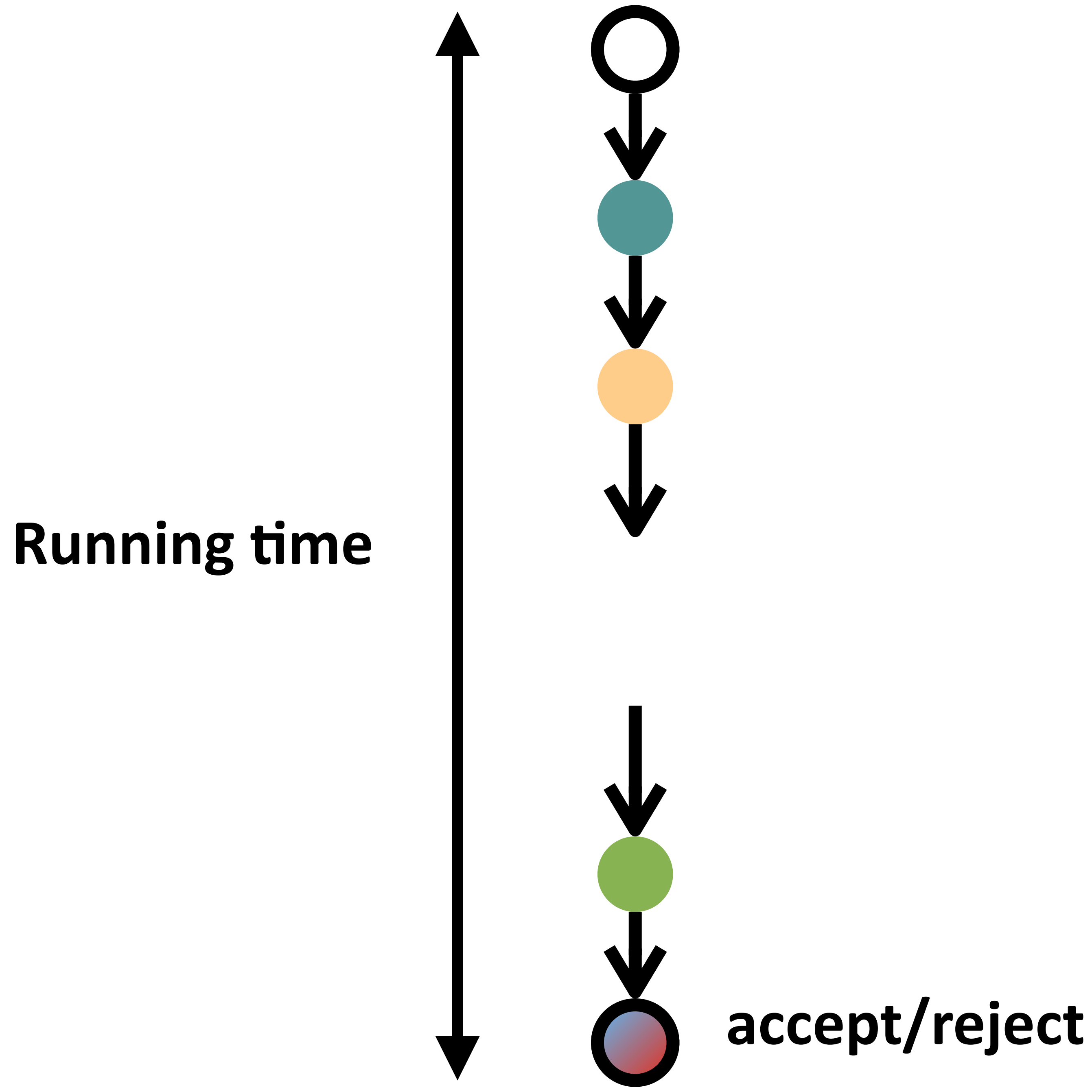
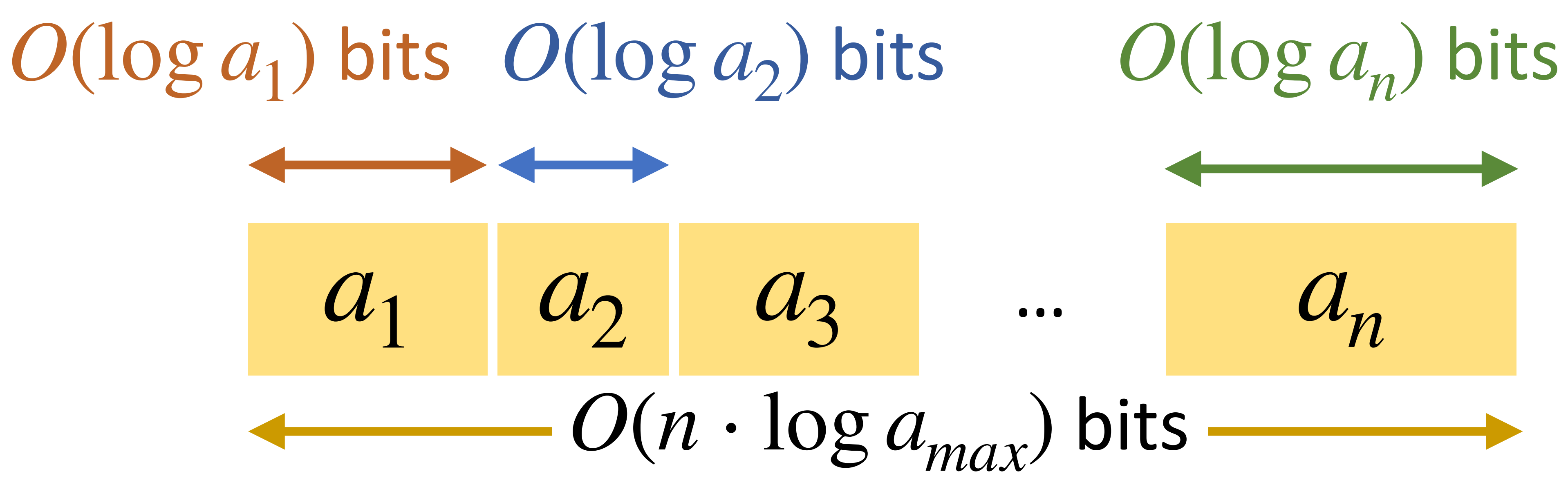
  - Ex: $f(n) = O(n^2)$
  - Ex: $f(n) = O(2^n)$





Input Size

- PARTITION $= \{\langle S = \{a_1, a_2, \cdots, a_n\}\rangle \mid S$ can be partitioned into two equal-sum subsets$\}$: a string w encoding the elements in S
- Input size: $O(n \log a_{max})$ bits
  - n: number of items in the set S
  - a_{max}: maximum value of the items in S. That is, for all $1 \leq i \leq n$, $a_i \leq a_{max}$.



Input Size

- CONNECT = $\{\langle G \rangle \mid G$ is connected graph$\}$: a string w encoding a graph G
- Input size: using binary encoding to encode vertices and edges in G
  - Use an adjacency array, the input size is $O(|V|\log|V| + |V|^2) = O(|V|^2)$ bits
  - Use an adjacency list, the input size is $O(|V|\log|V| + |E| \cdot 2\log|V|) = O(|V|^2\log|V|)$ bits
  - The $O(\log|V|)$ bits are for encoding the vertex ID

$v_1, v_2, v_3, \cdots, v_n$	$e_1 = (v_{i_1}, v_{j_1}), e_2 = (v_{i_2}, v_{j_2}), \cdots, e_m = (v_{i_m}, v_{j_m})$
$\longleftarrow O(|V|\log|V|) \longrightarrow$	$\longleftarrow O(|E| \cdot 2\log|V|) \longrightarrow$

Input Size

- PRIME = {prime number}: a string w representing number n
- Input size?

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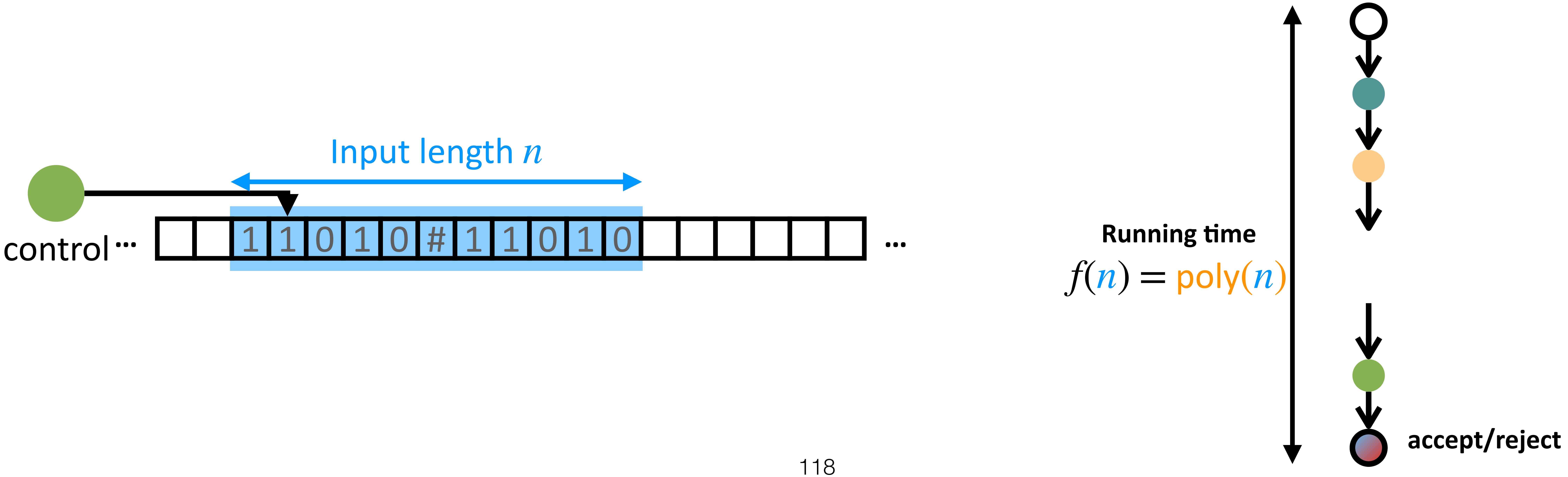
The Class **P**

- Definition: **P** is the class of languages that are can be accepted or rejected in polynomial time by a deterministic single-tape Turing machine.



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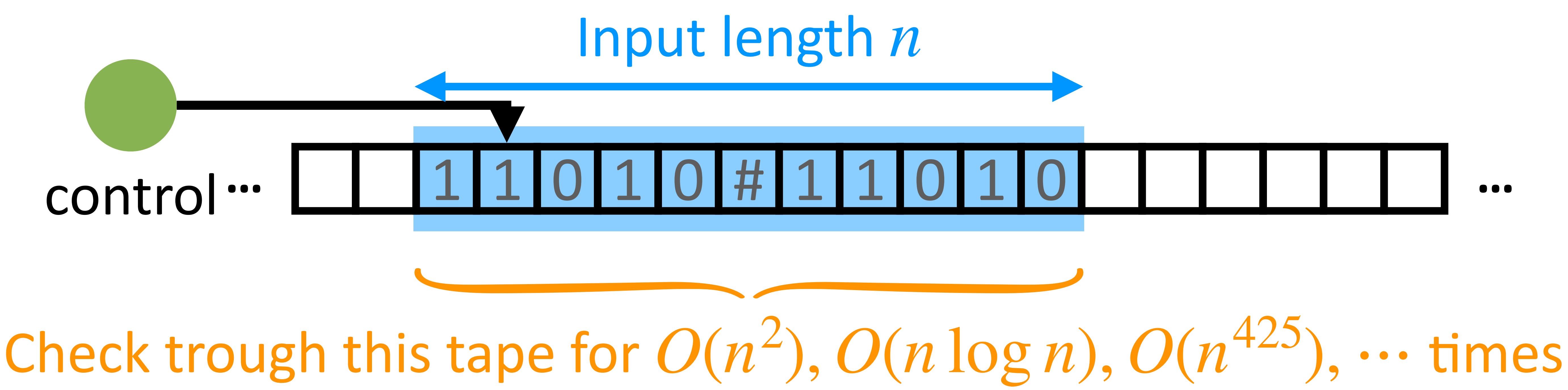

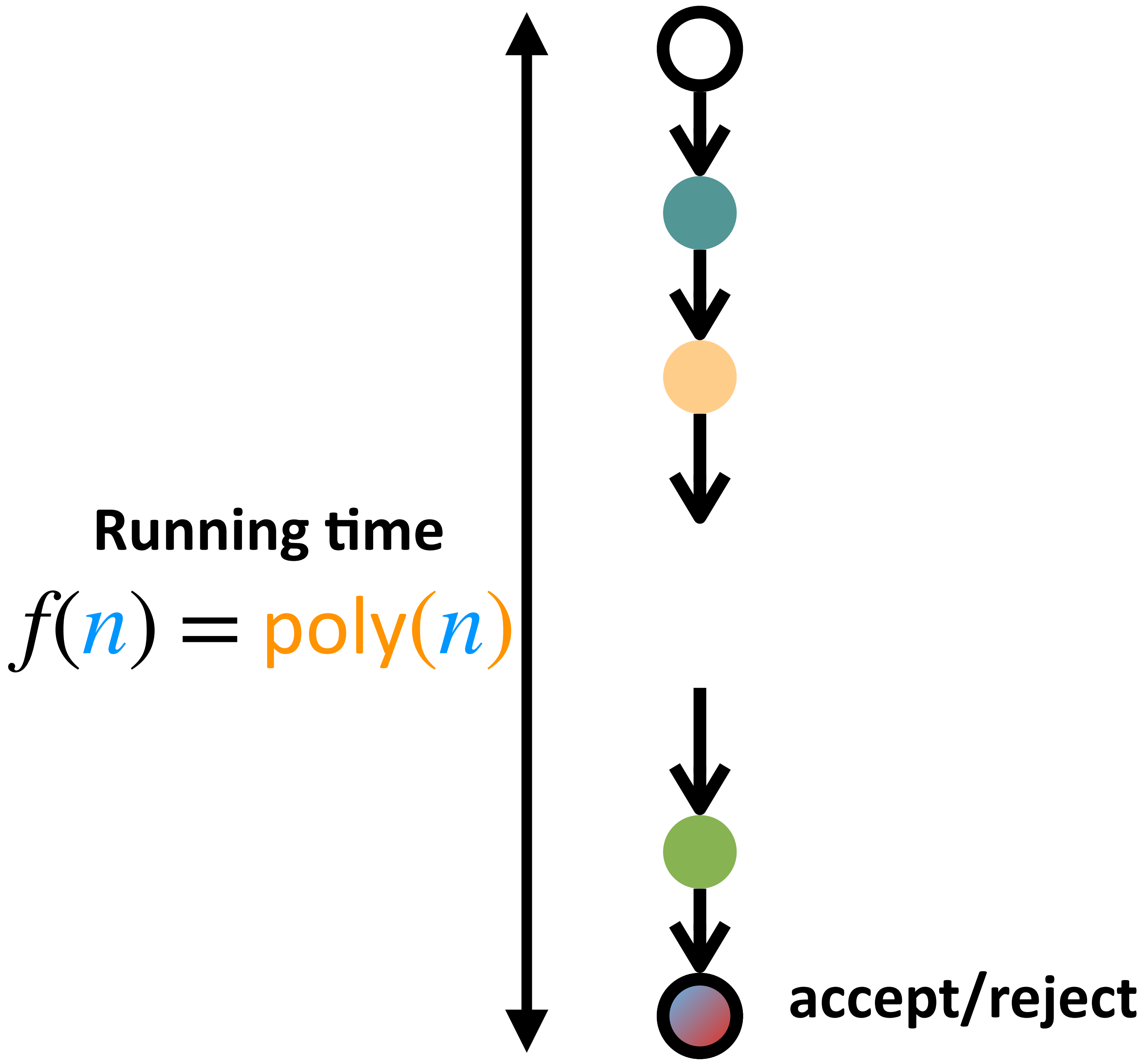

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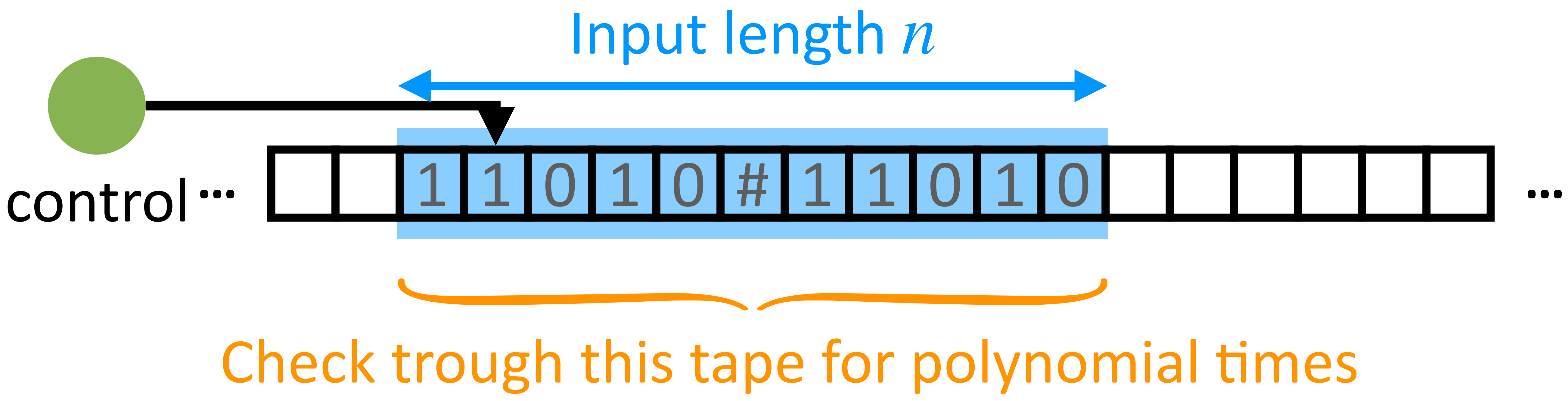
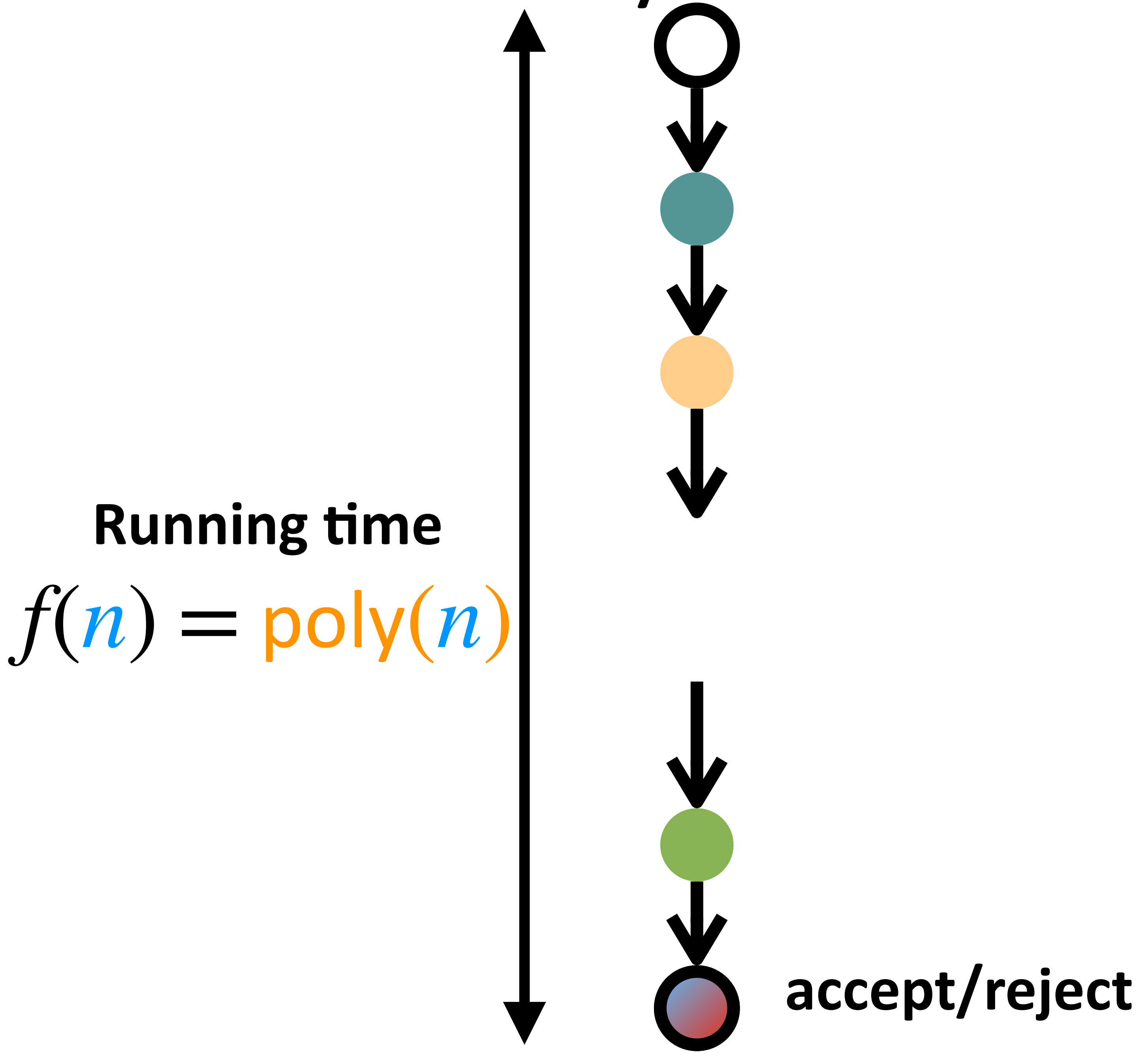
  - **P** roughly corresponds to the class of problems that are realistically solvable on a computer





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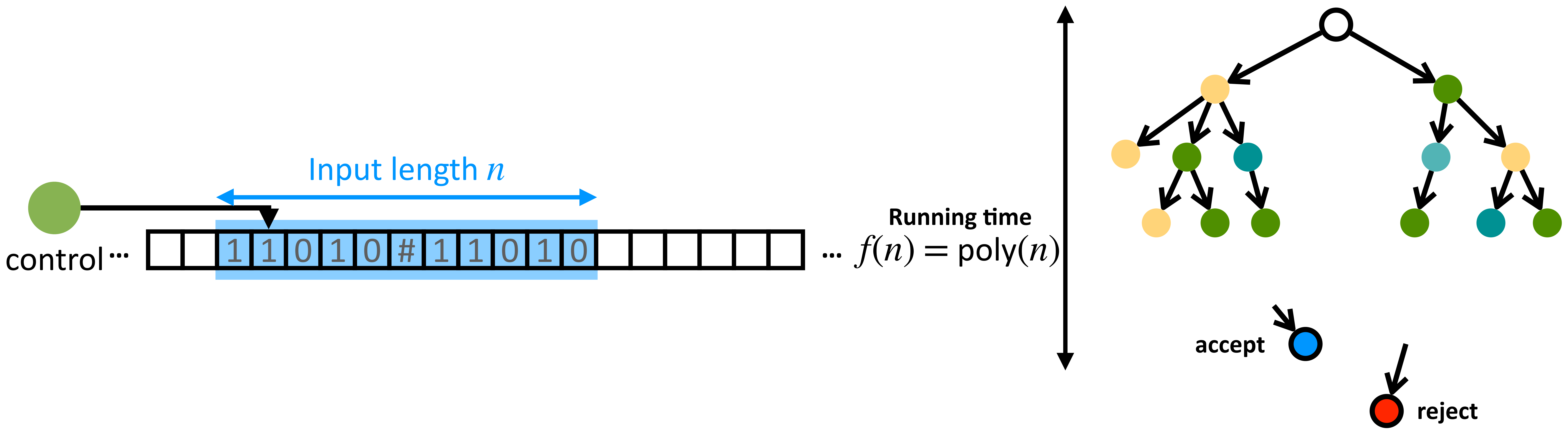

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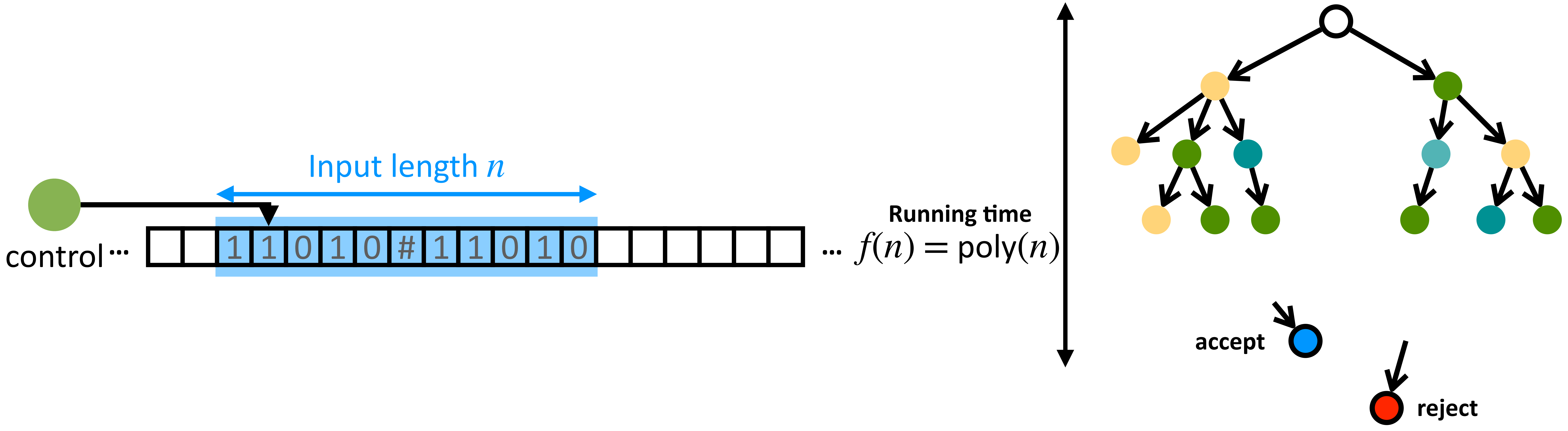

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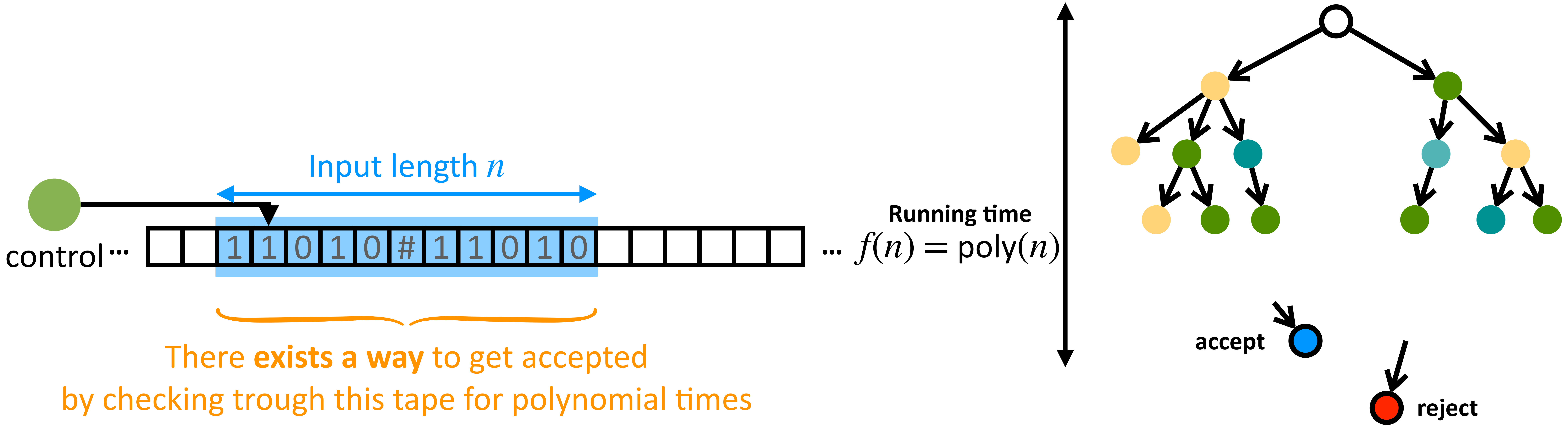
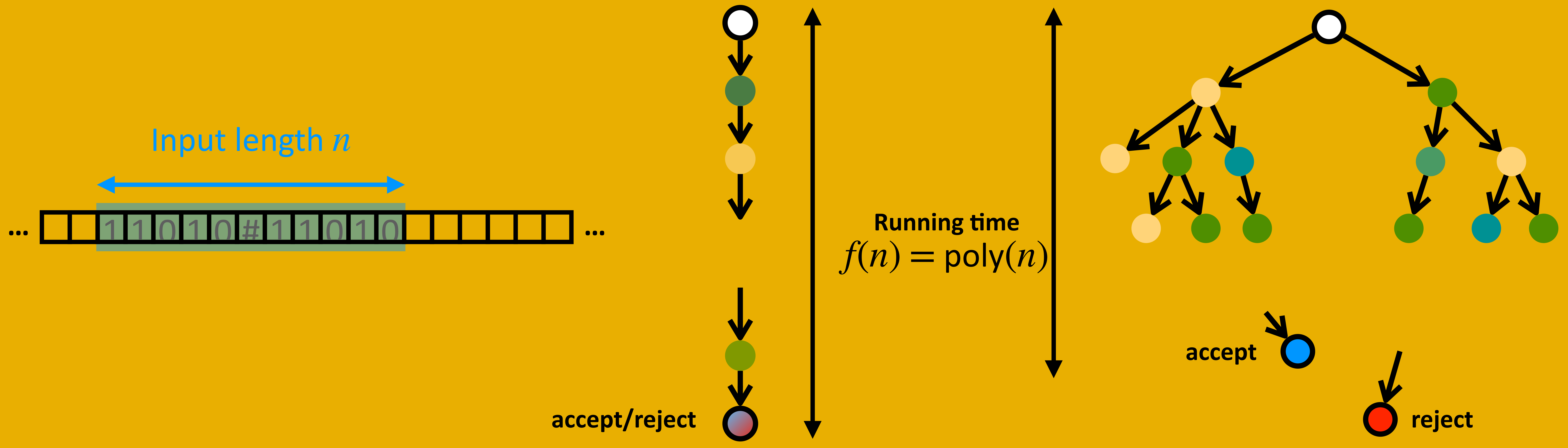


What Happened

- The class **P** is the class of languages that are accepted or rejected in polynomial time by a *deterministic* Turing machine
- The class **NP** is the class of languages that are accepted in polynomial time by a *non-deterministic* Turing machine.



Overview

- Algorithms:Turing machine, Deterministic and non-deterministic
- Formal language framework: string and language
- Time complexity
  - Input size
  - Classes **P** and **NP**
    - Polynomial time verification

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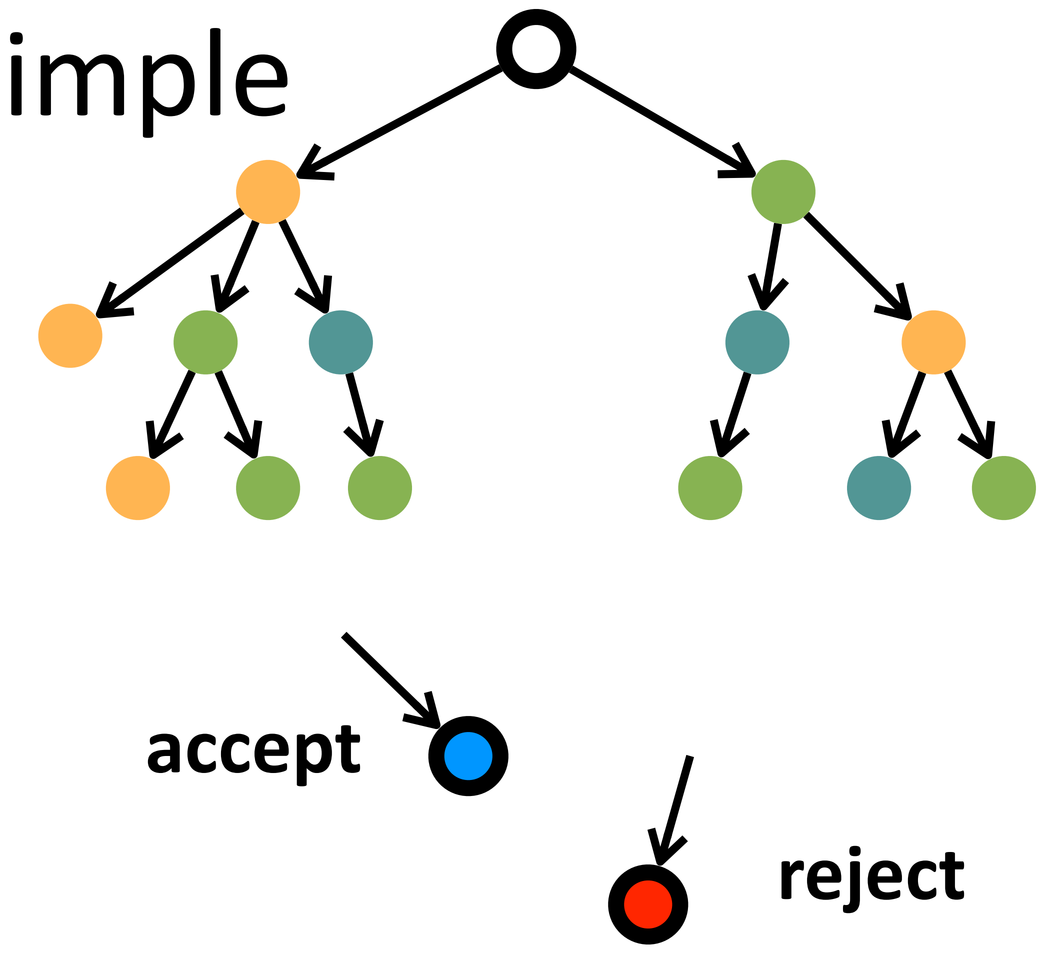

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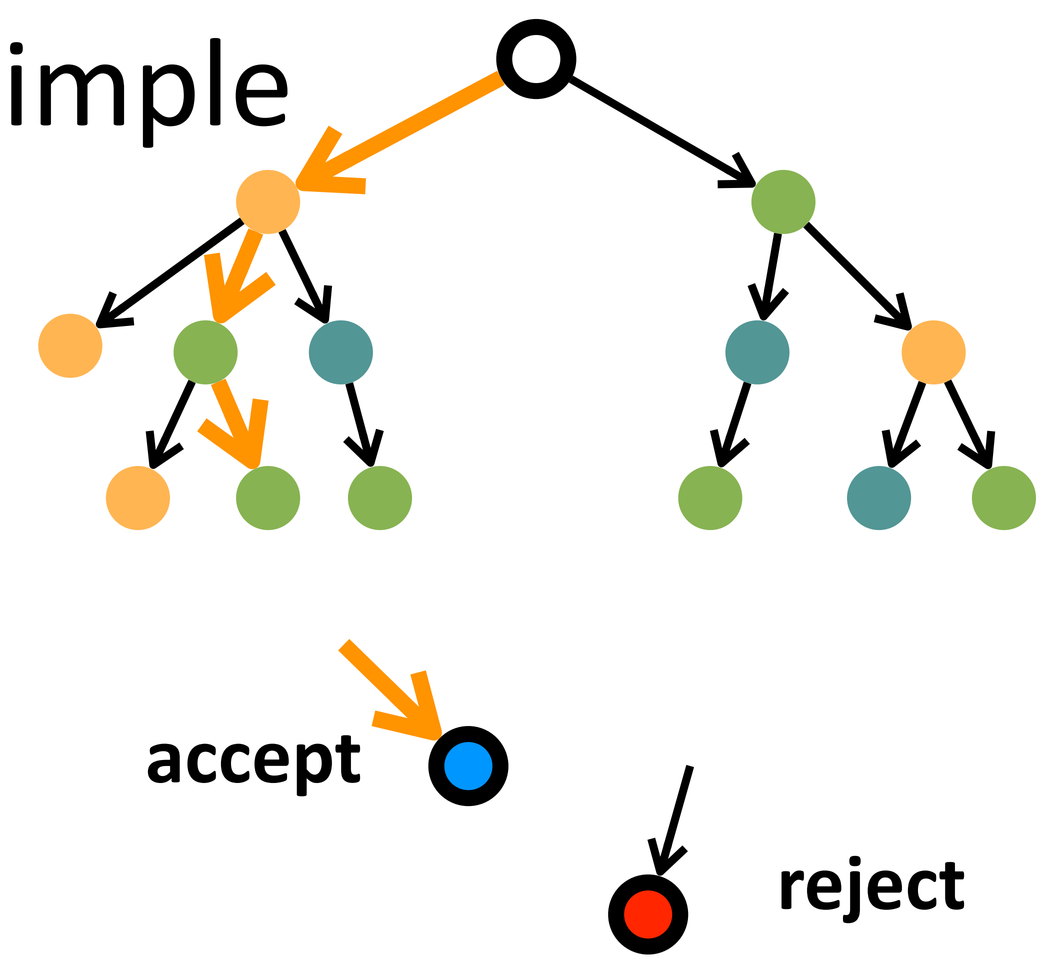

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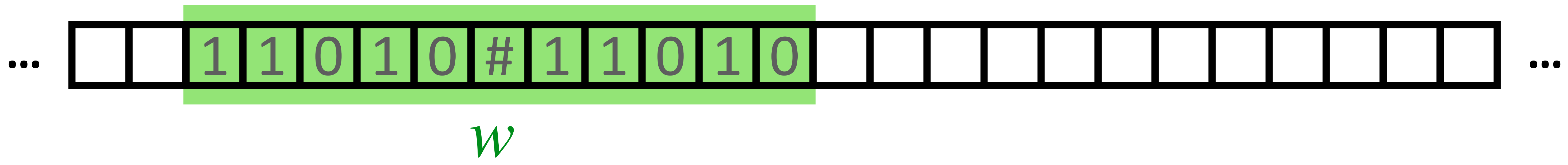

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A set of k vertices with edge between every pair of vertices

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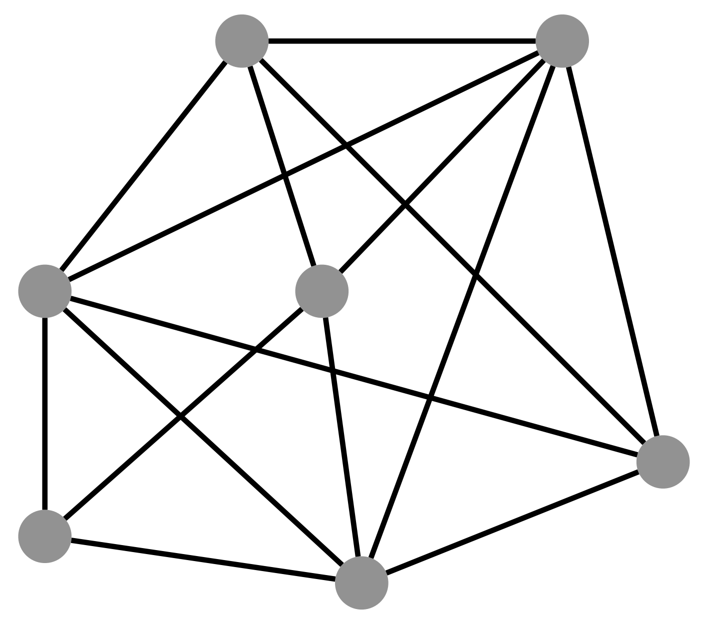

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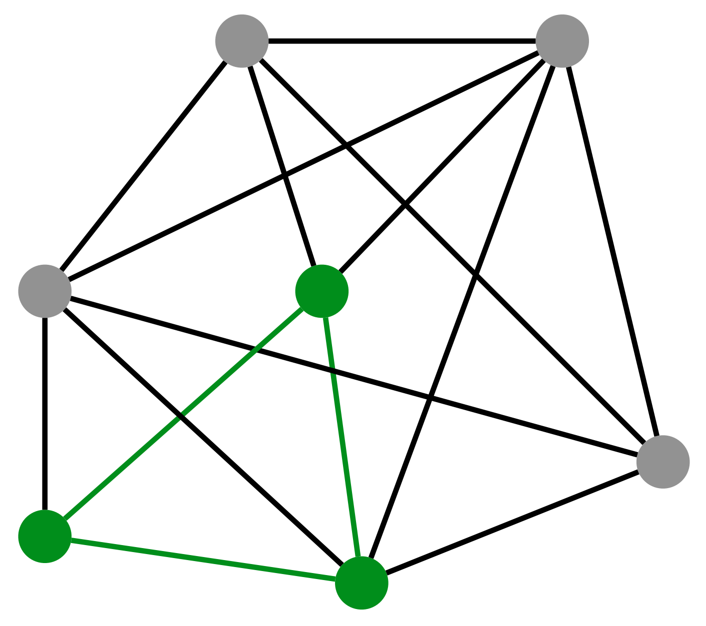

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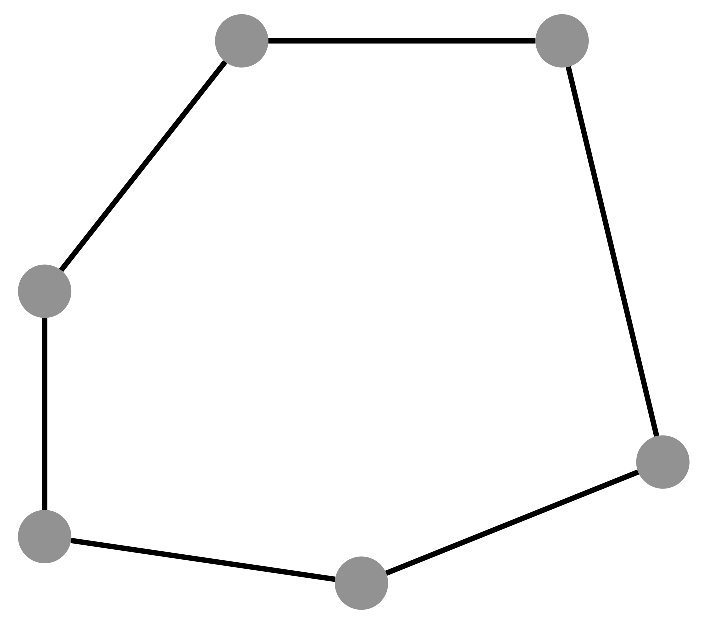

❌ $\langle G,3 \rangle$ is a no-instance

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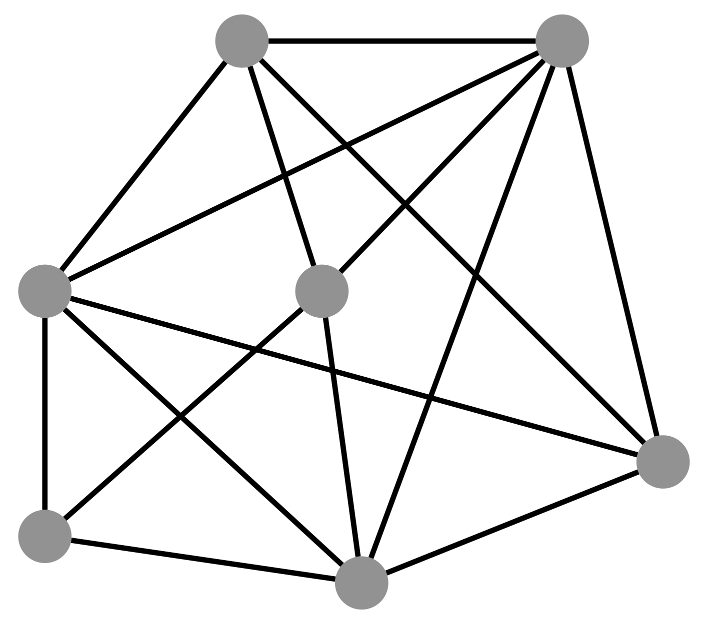

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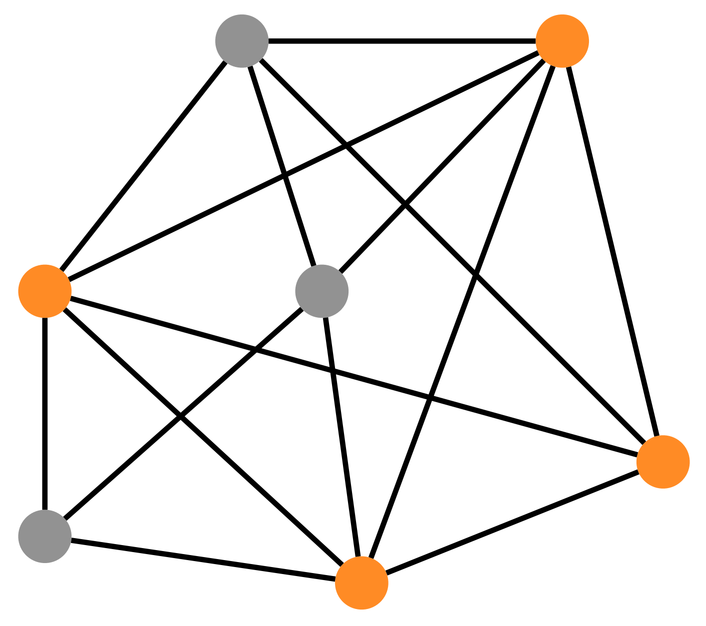

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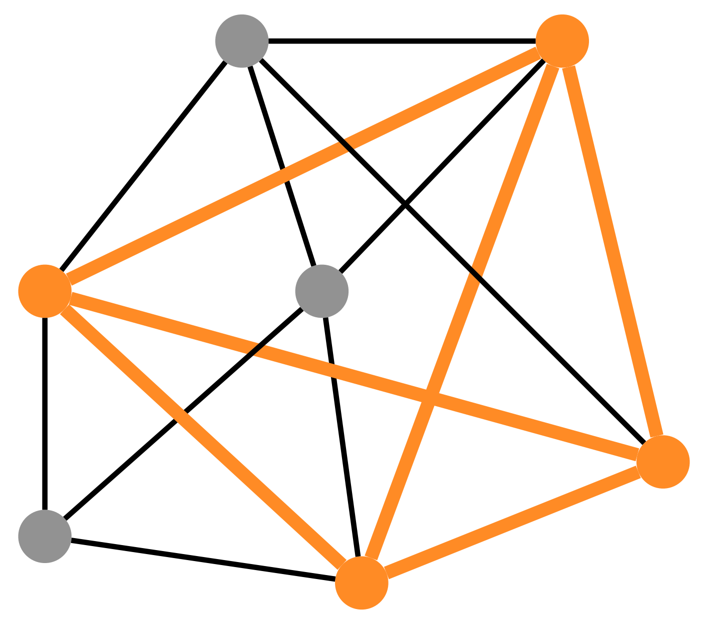

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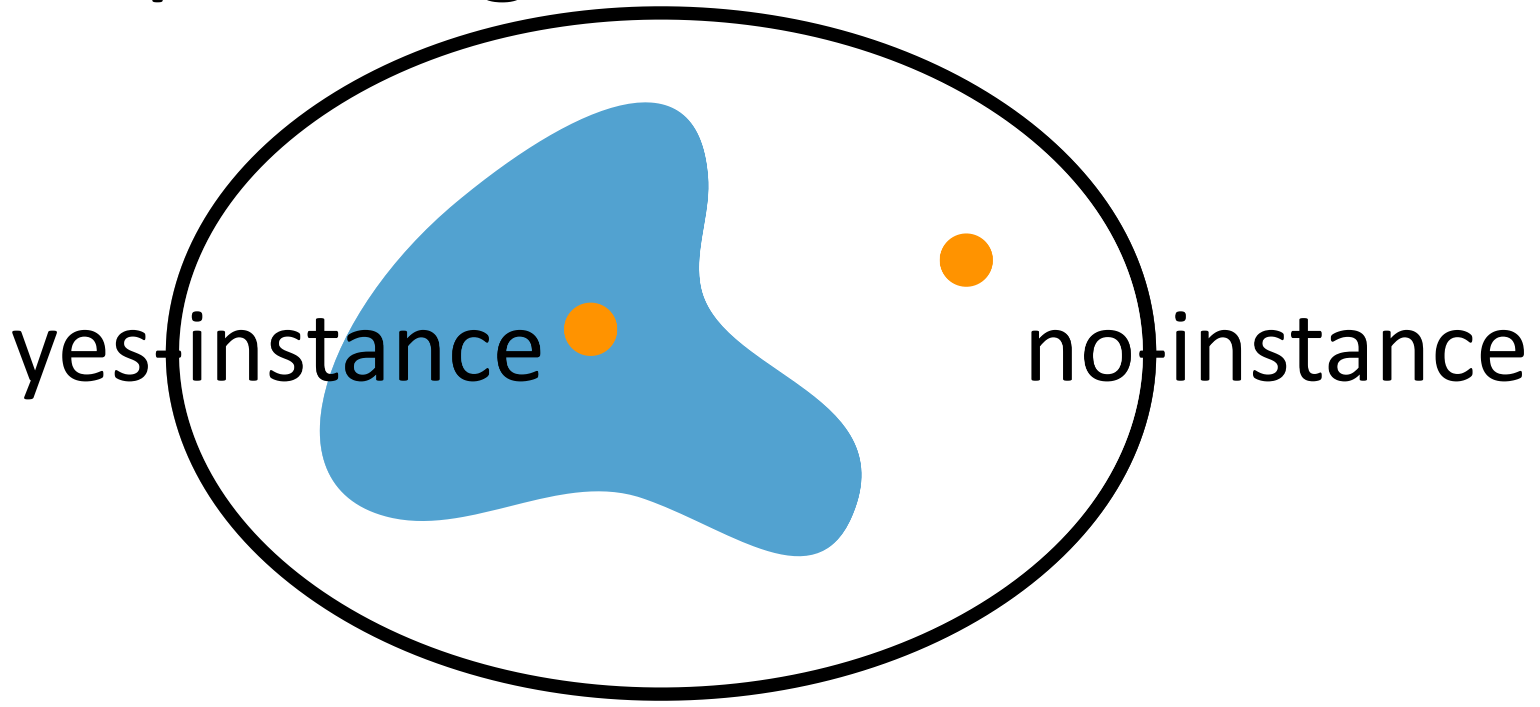

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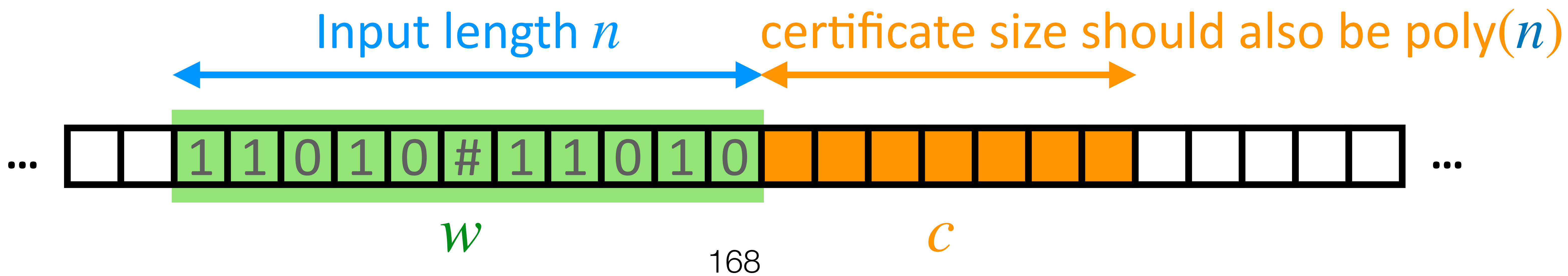

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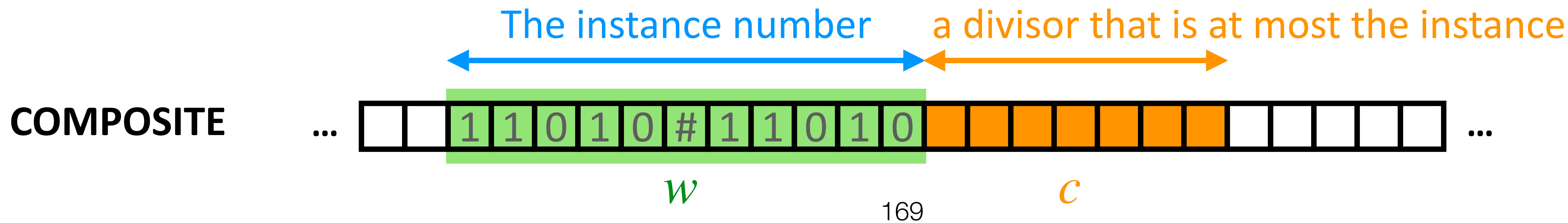

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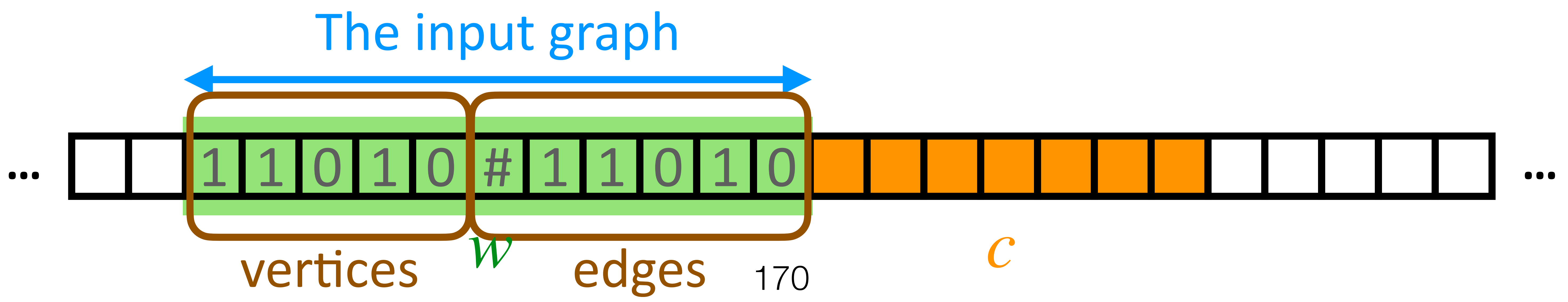
CLIQUE



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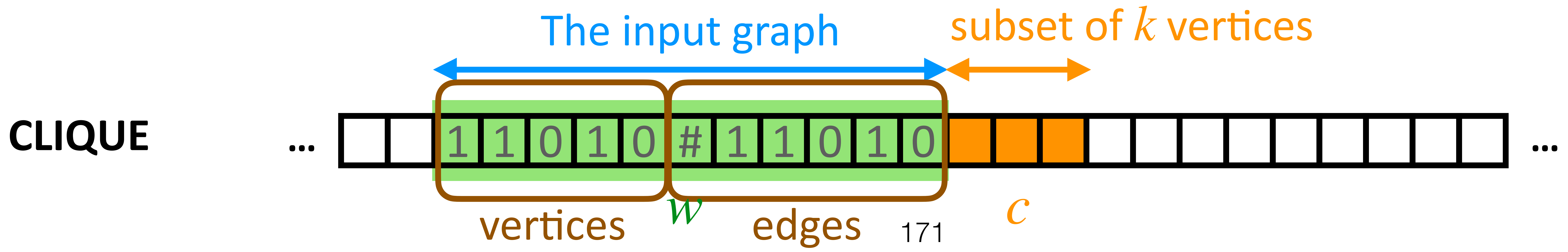

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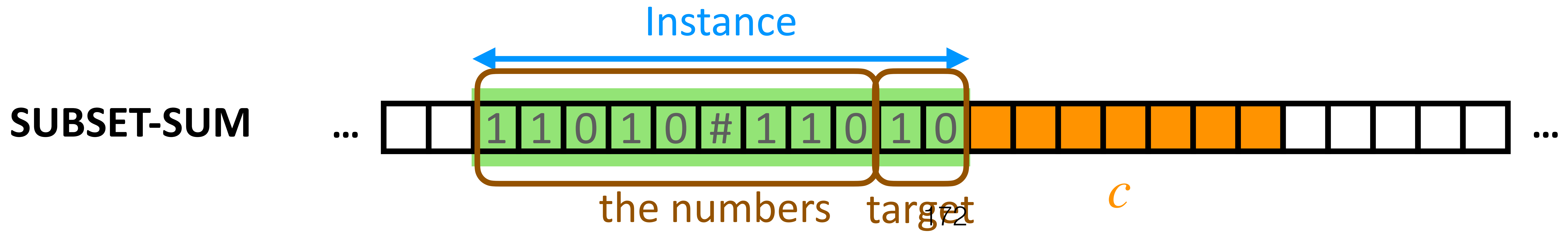

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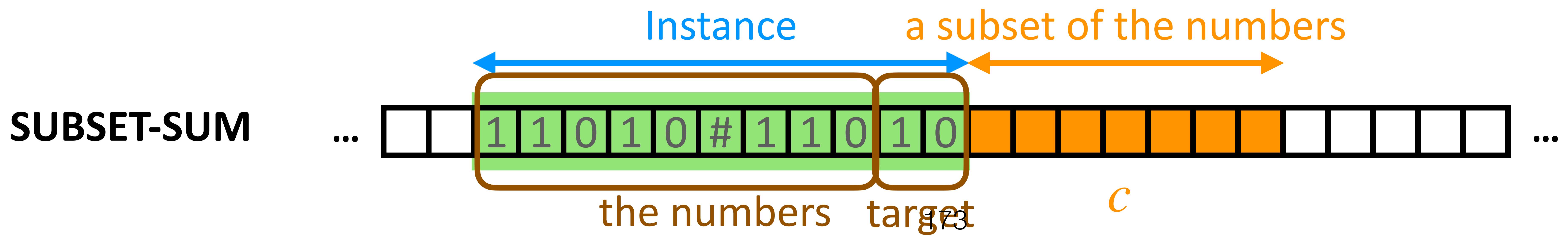
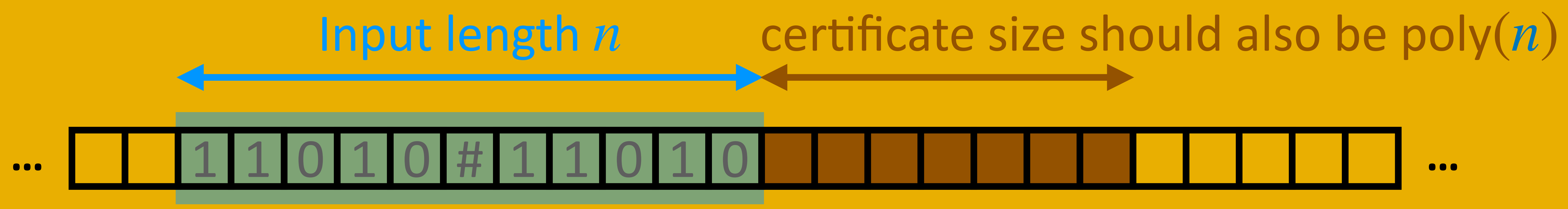


What Happened

- A language A is **verifiable** if for any of its yes-instances w, there exists a piece of hint (certificate) c such that using this hint c, one can be convinced that w is indeed a yes-instance of A
  - Only yes-instances have certificates
- **Polynomial-time verifiable**: the verification can be done in time of polynomial in input length
  - The hint size should also be polynomial
  - It does NOT mean that the hint c should be constructed within polynomial time!



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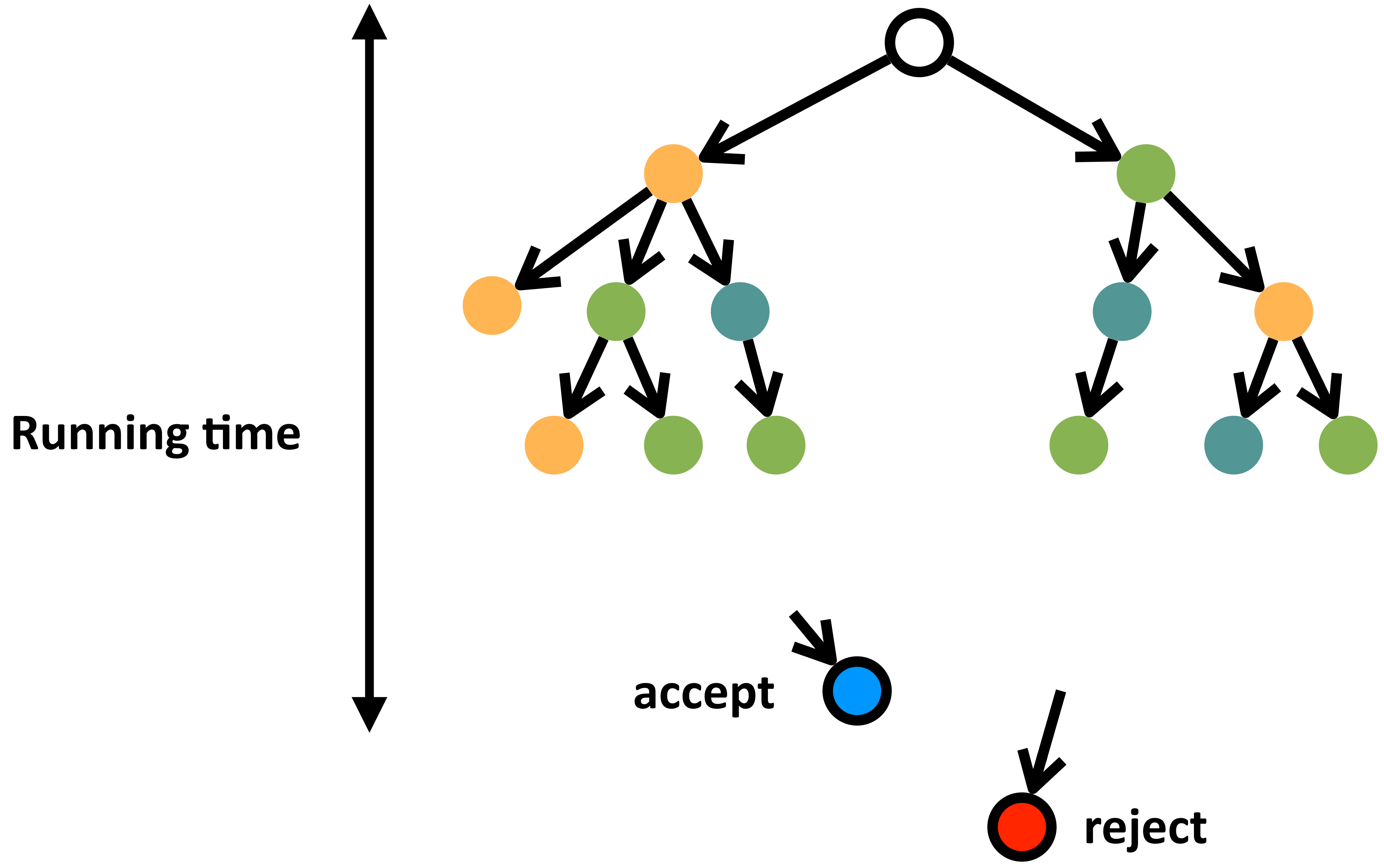
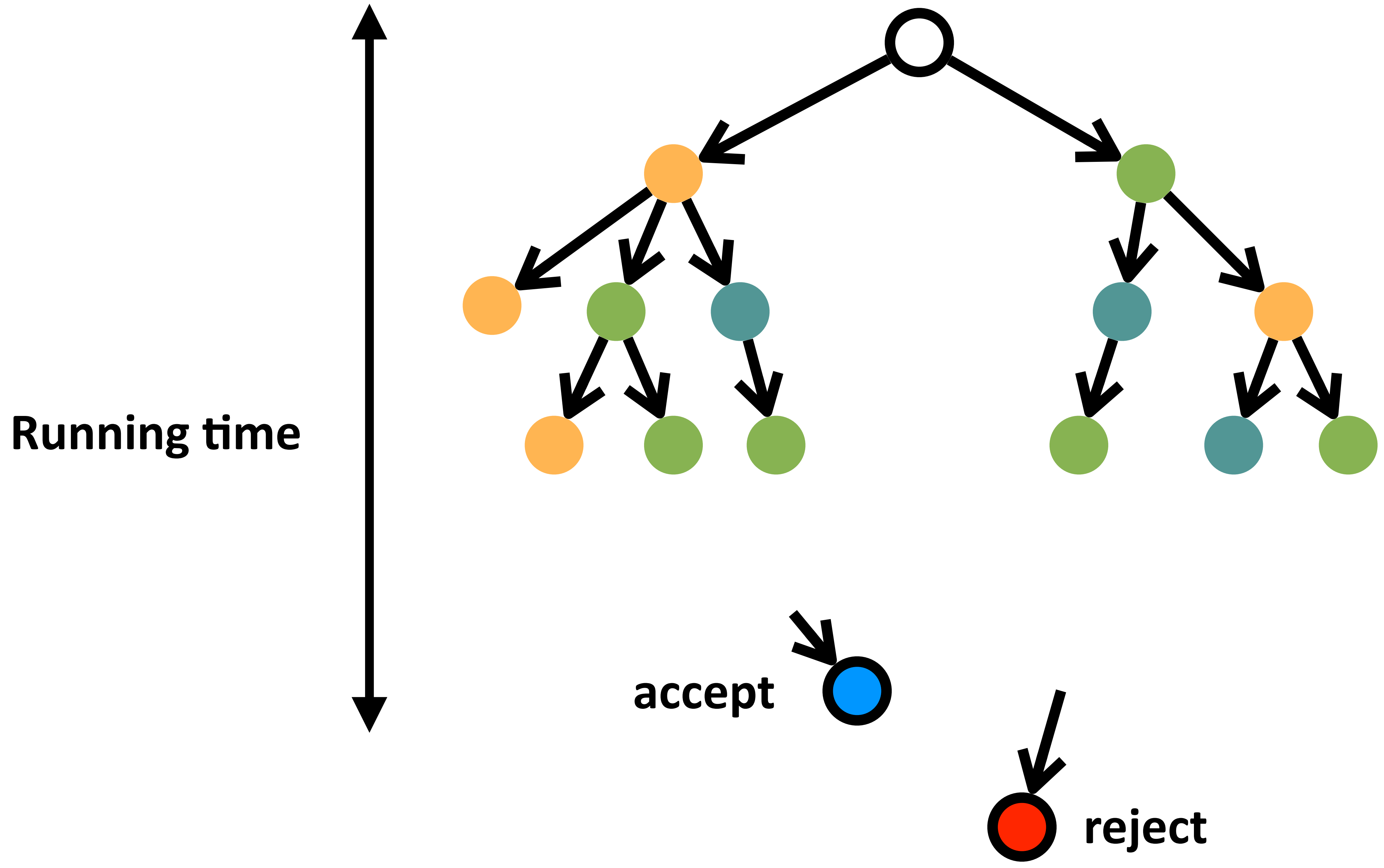


The Class **NP** — Alternative Definition

- Definition: **NP** is the class of languages that are verifiable in polynomial time on a deterministic Turing machine.



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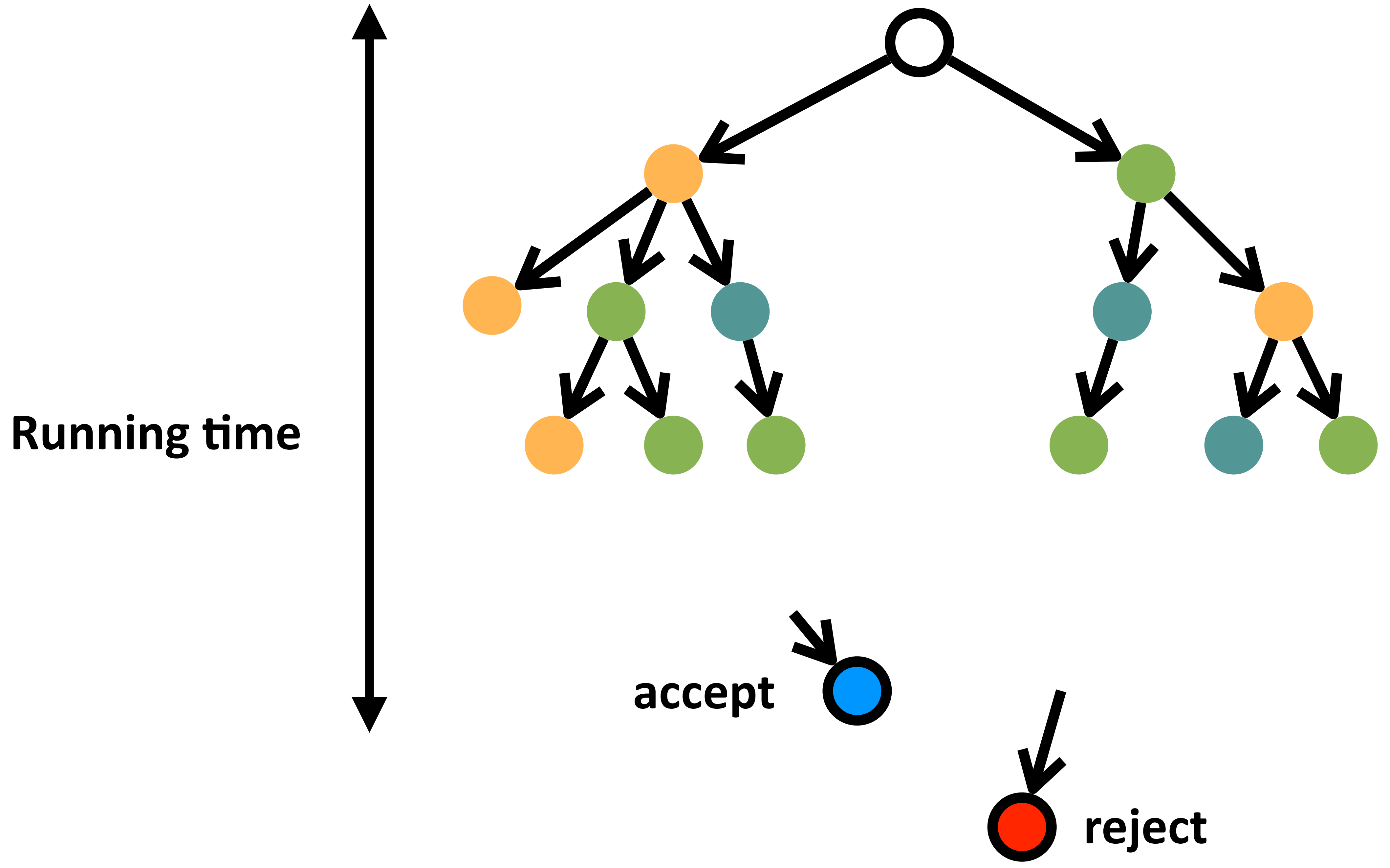

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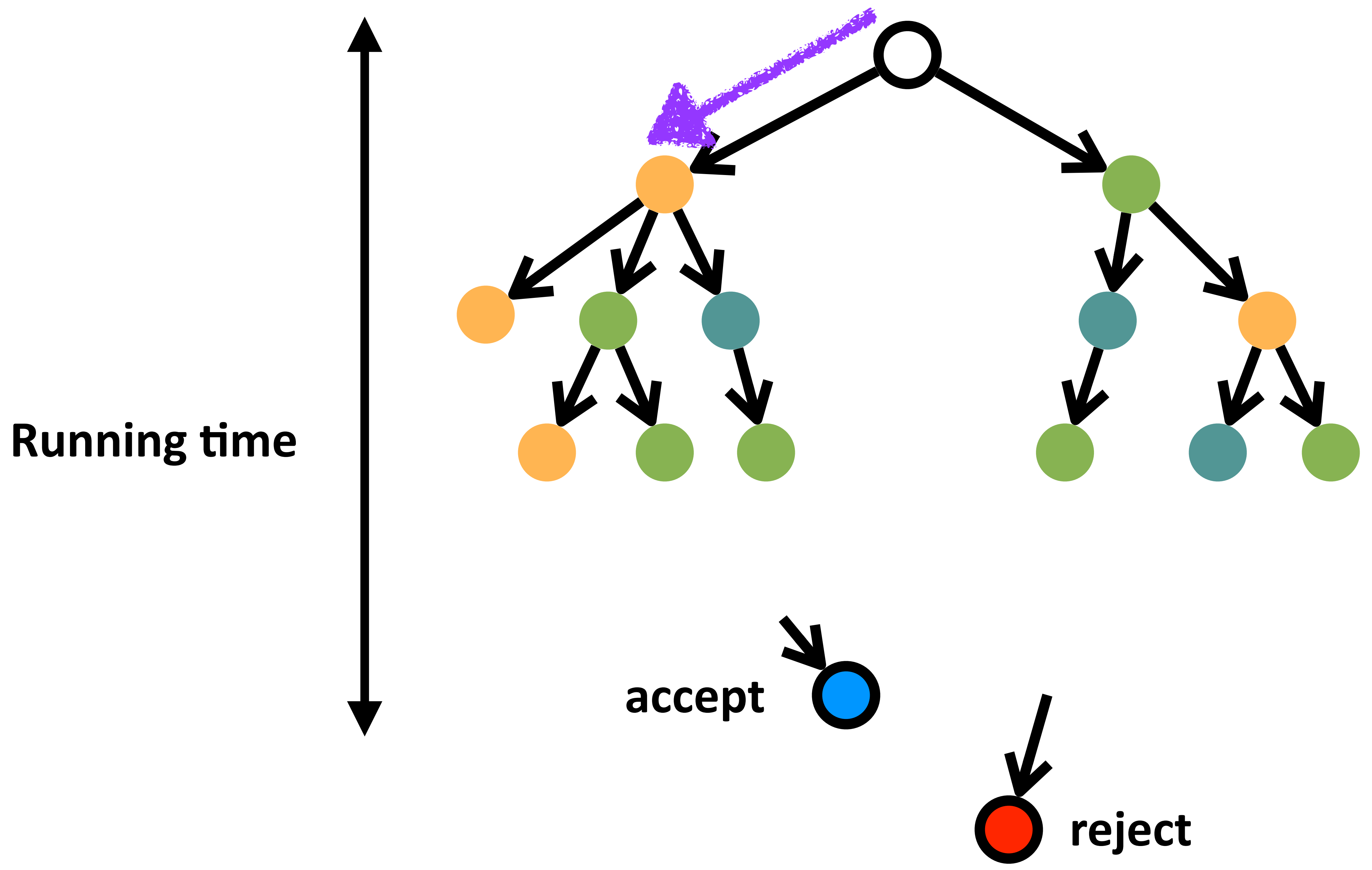

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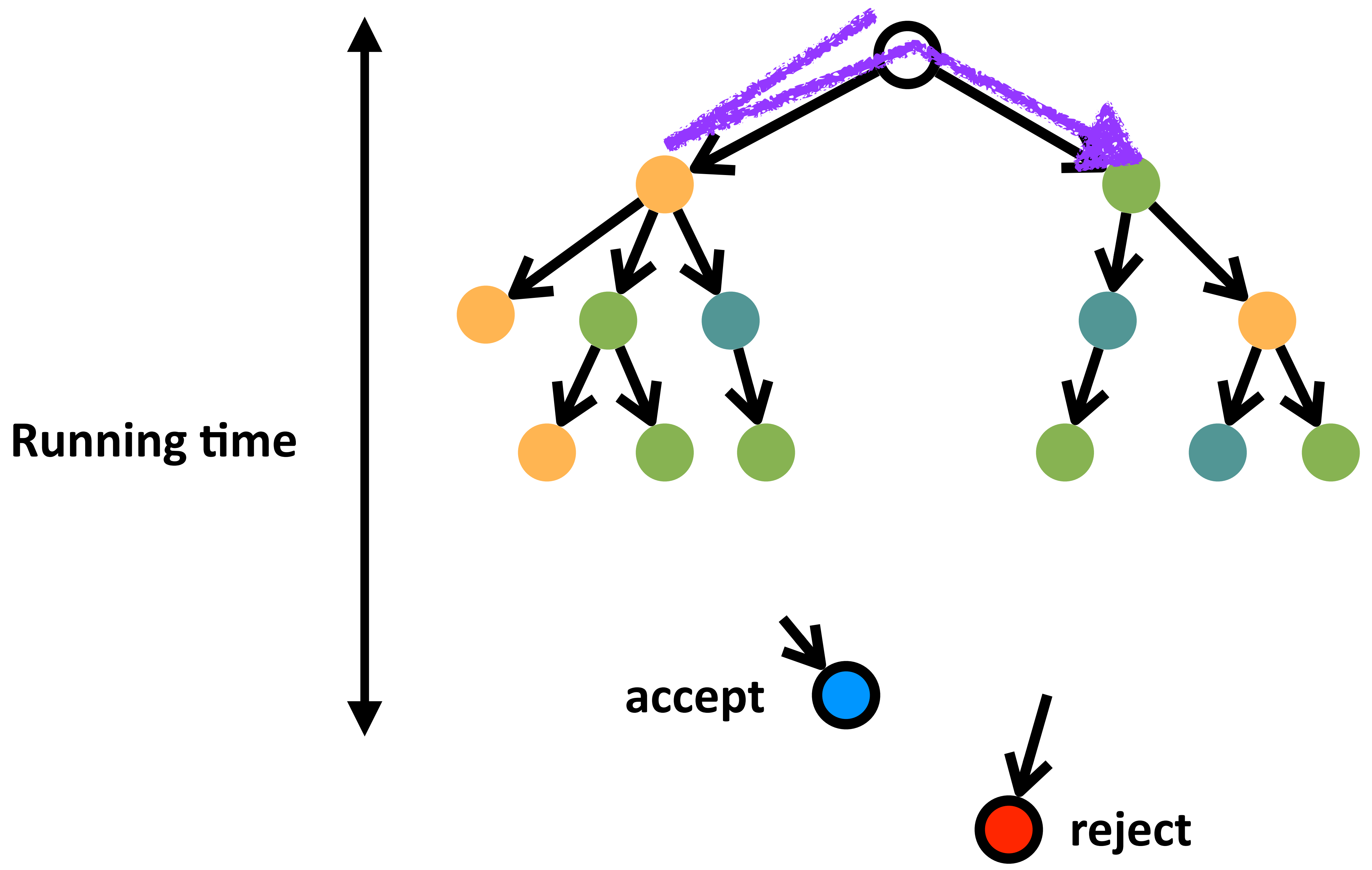

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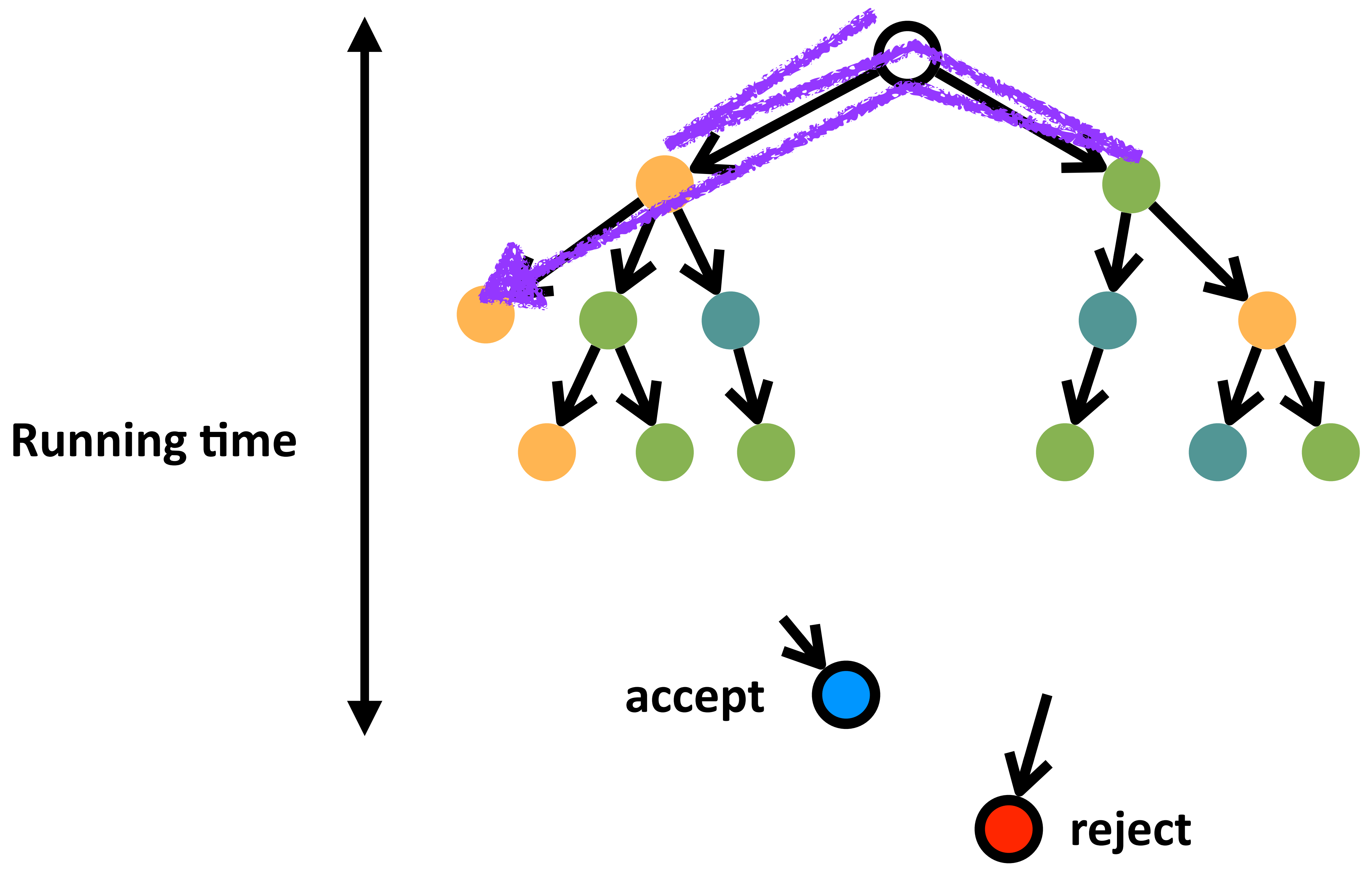

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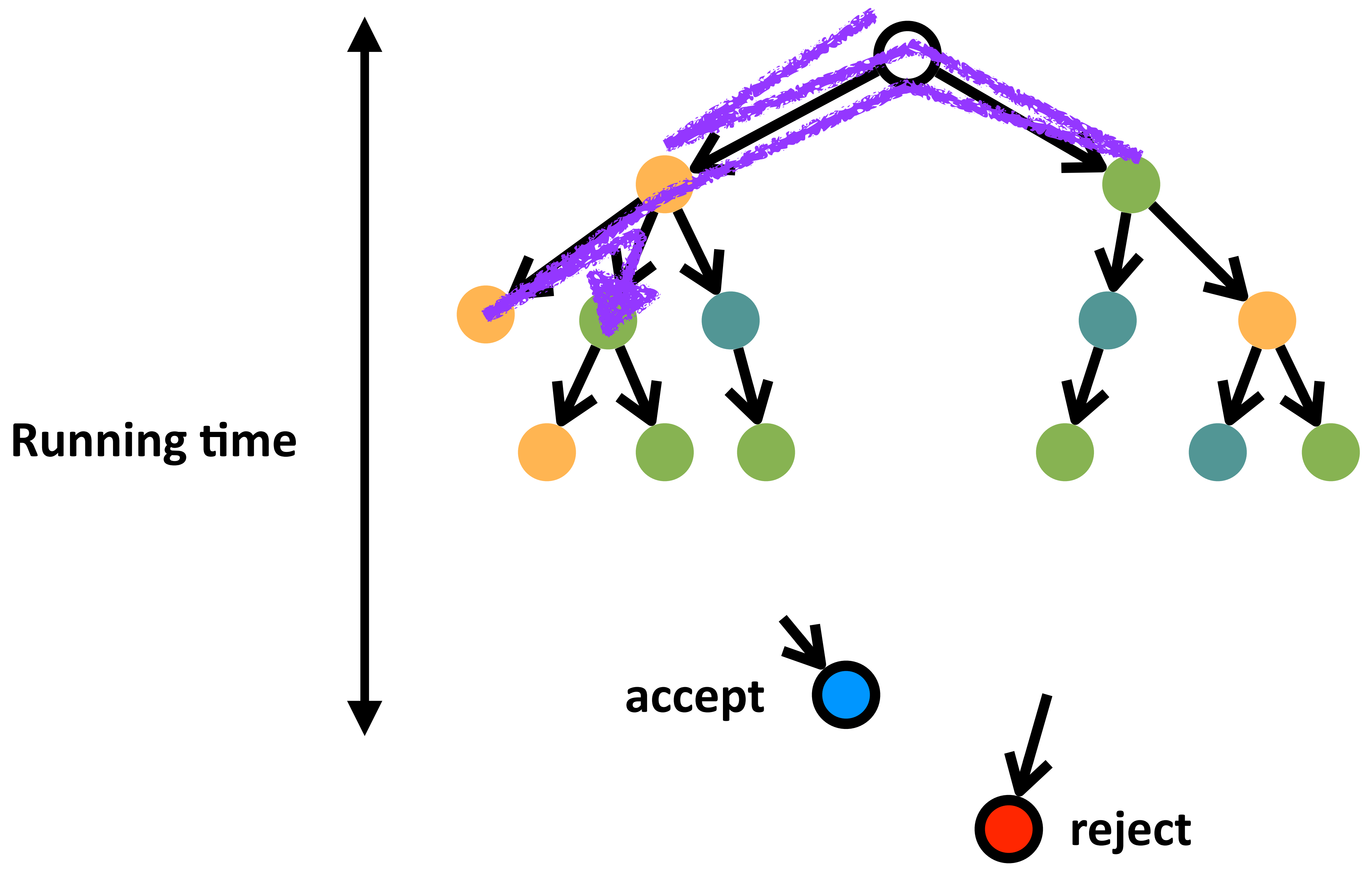

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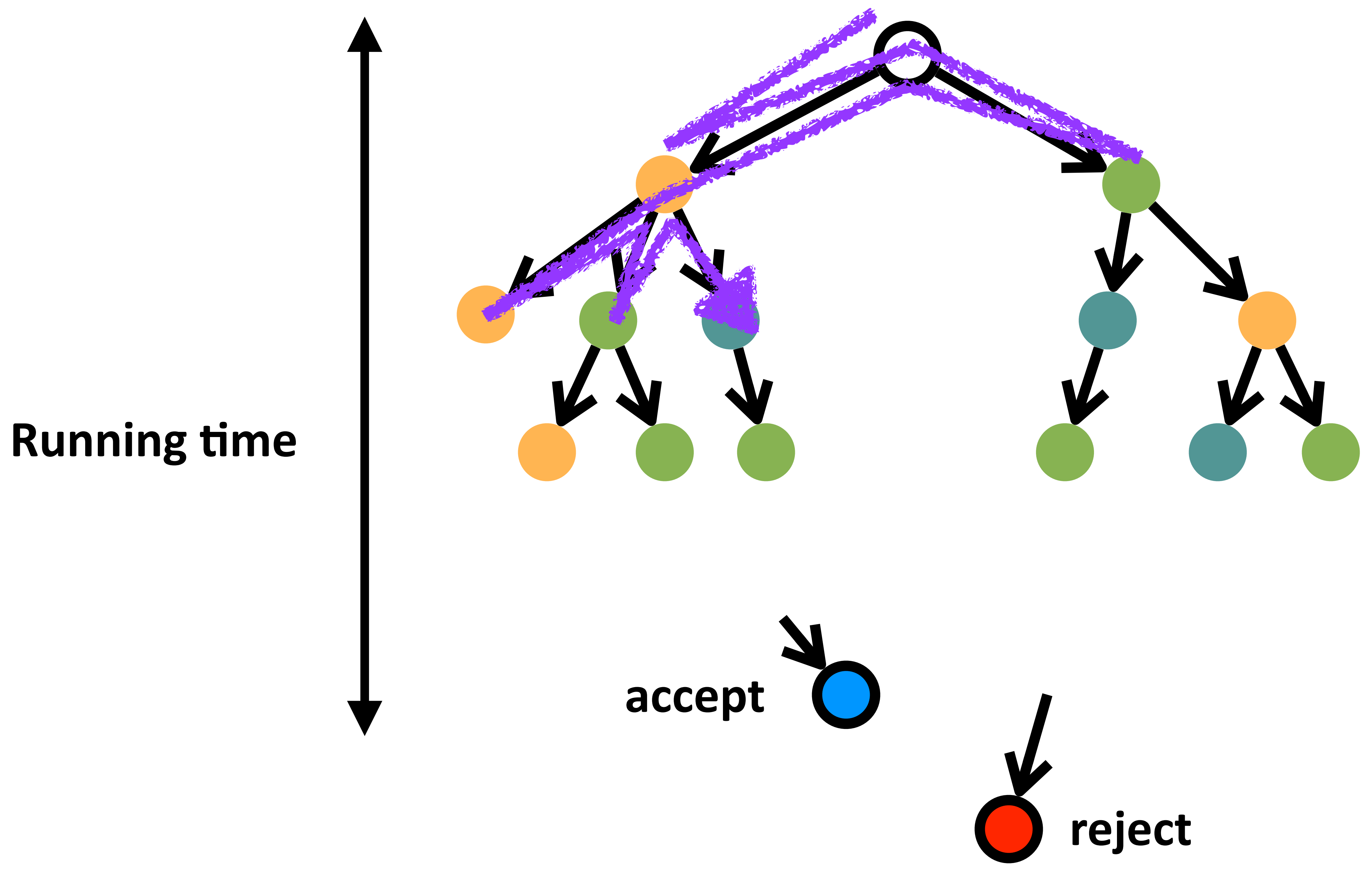

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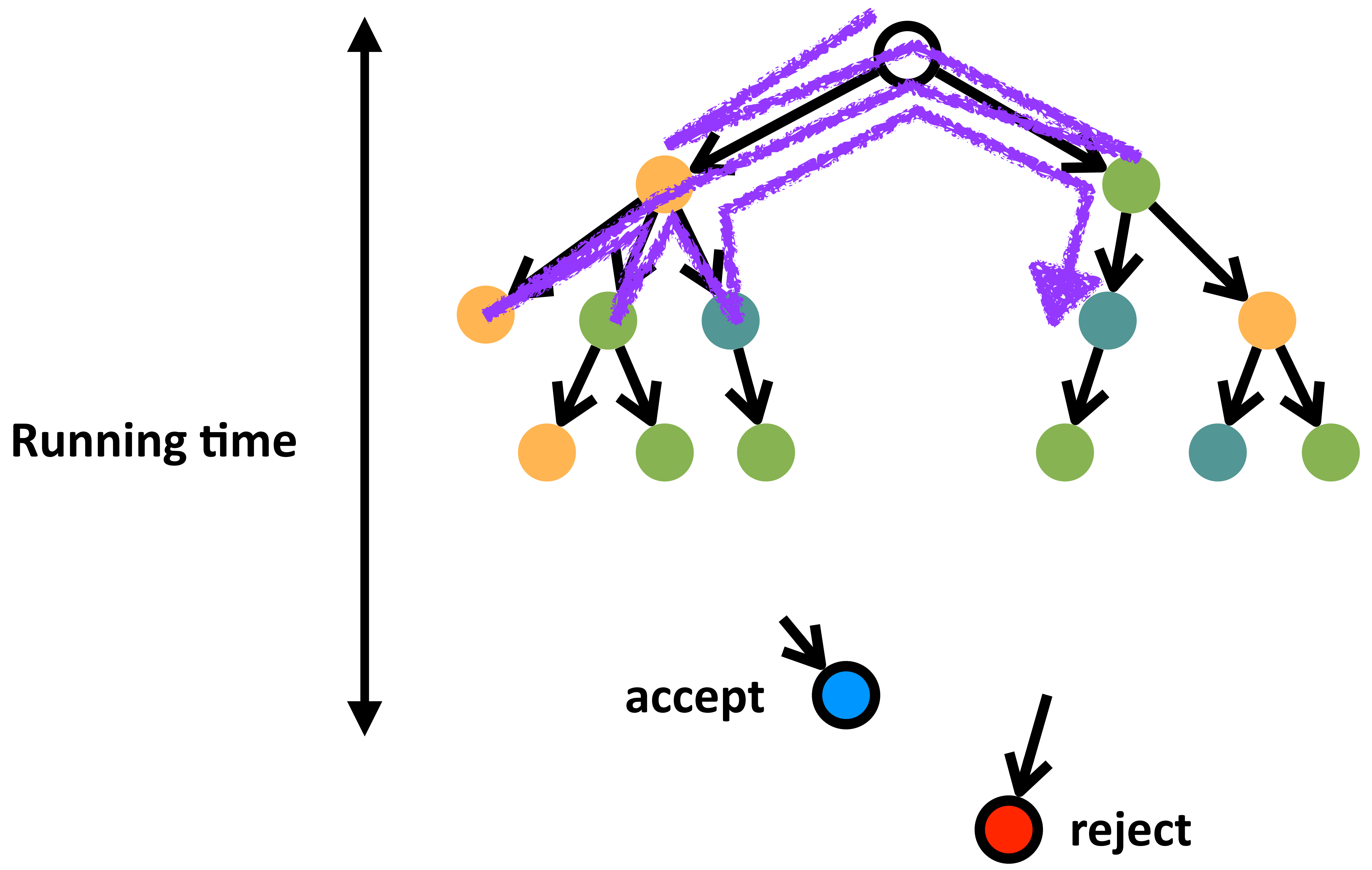

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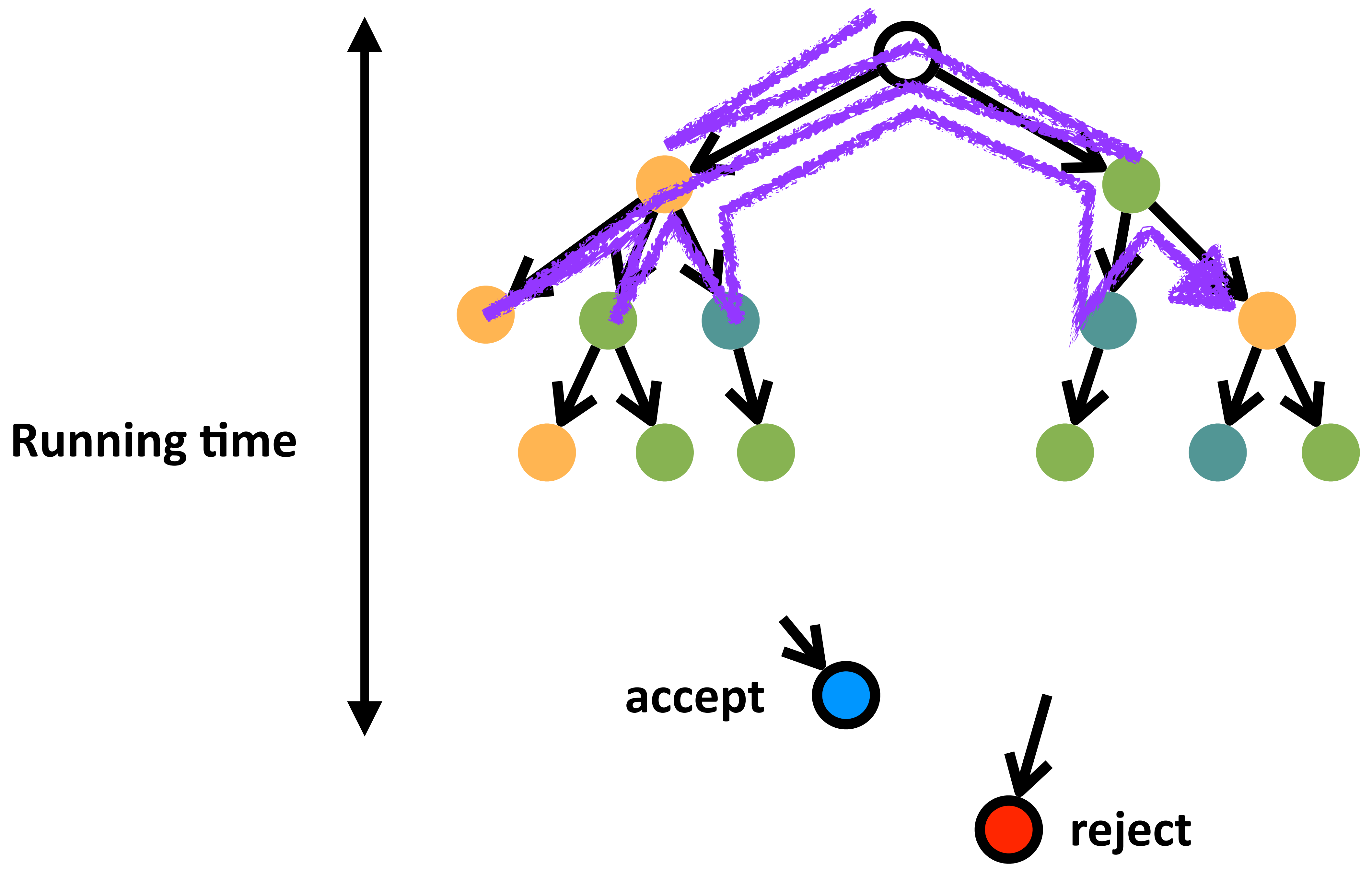

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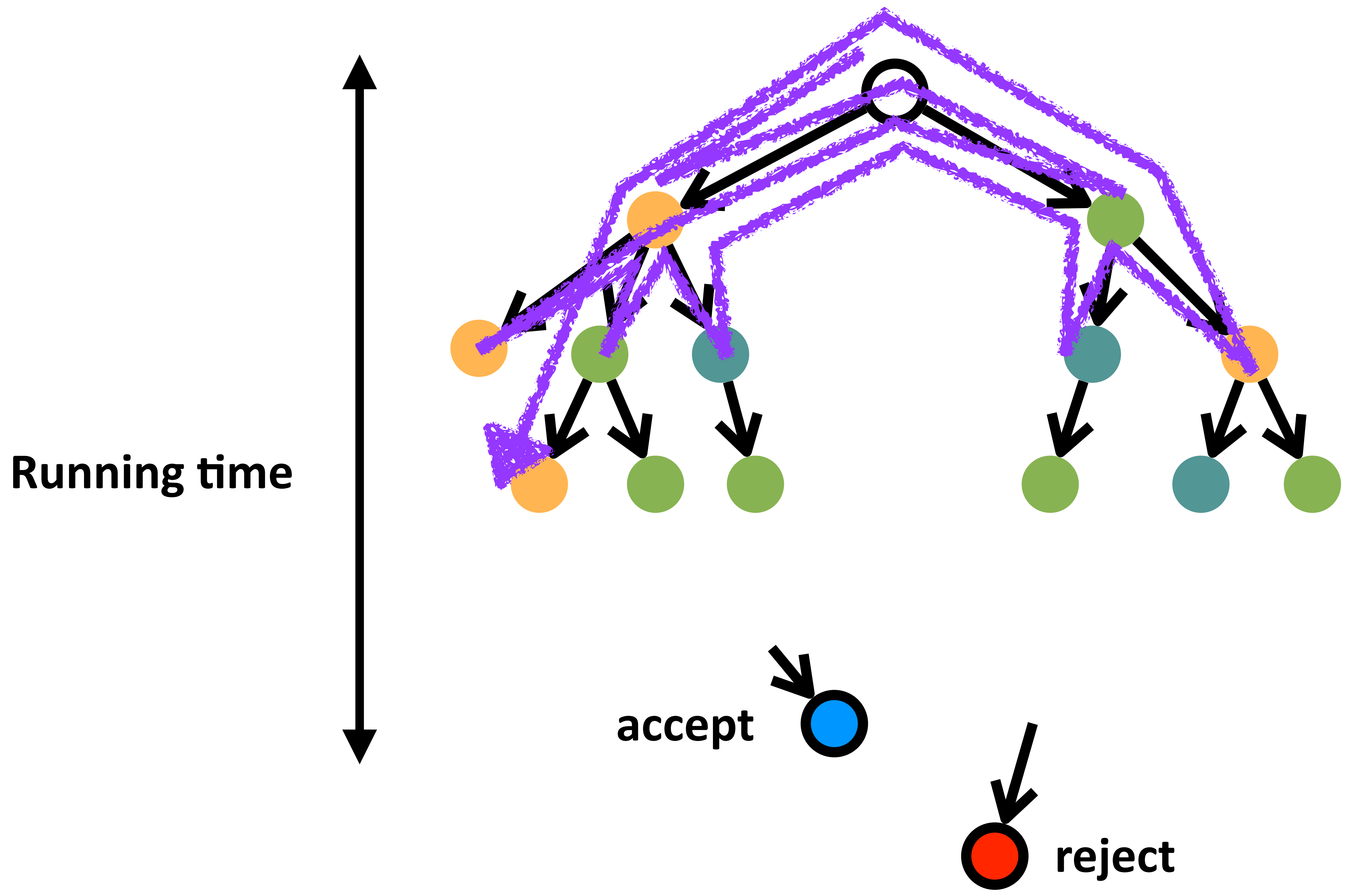

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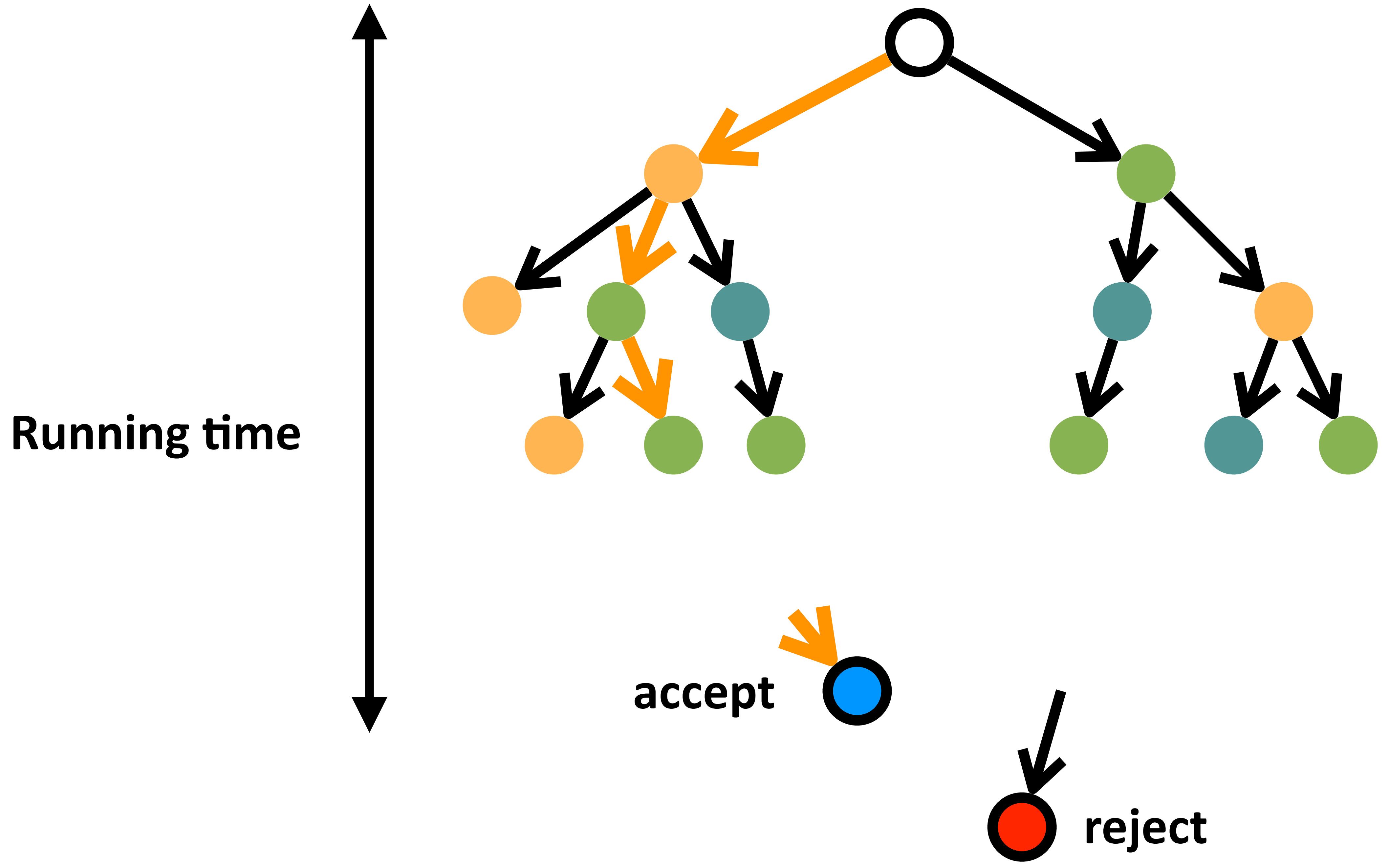
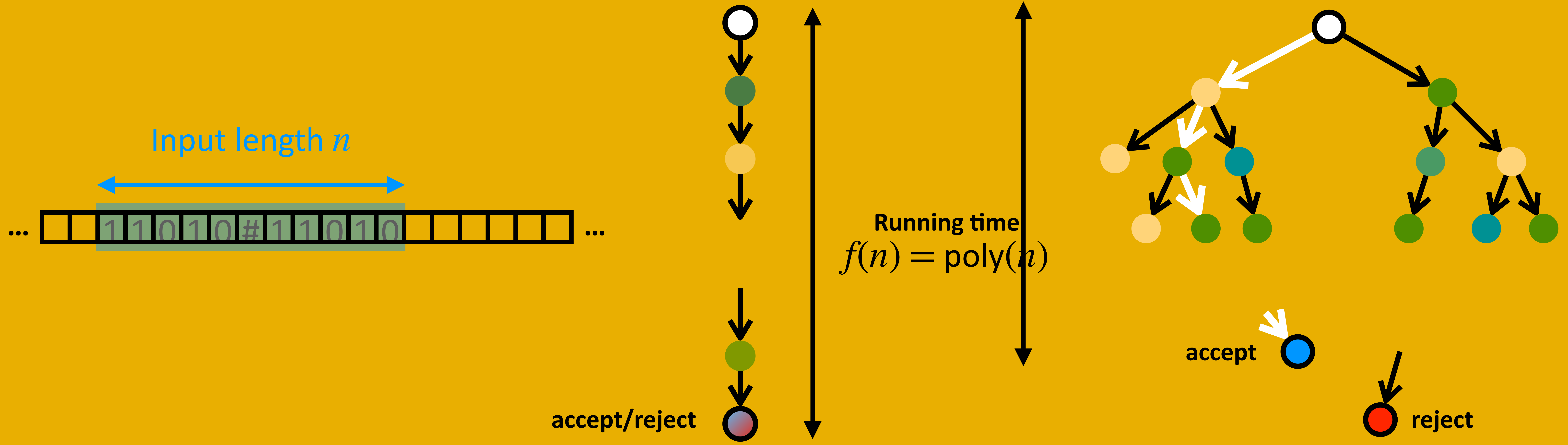
- Definition: **NP** is the class of languages that are verifiable in polynomial time on a deterministic Turing machine.
- Any non-deterministic Turing machine can be simulated by a deterministic Turing machine
  - It needs more than polynomial time
  - But if we know some **hint**, we know the path to an accept state with polynomial length



What Happened

- The class **P** is the class of languages that are *accepted* or *rejected* in polynomial time by a deterministic Turing machine
- The class **NP** is the class of languages that can be verified in polynomial time by a deterministic Turing machine.



Prove Language L is in **P**

- To prove that a language L is in **P**, we need to:
  - Design a Turing machine M
  - Show that M correctly accepts or rejects all input
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Design an algorithm

Correctness proof

Time complexity analysis

Prove Language L is in **P**

- To prove that a language L is in **P**, we need to:
  - Design a Turing machine M — Design an algorithm
  - Show that M correctly accepts or rejects all input — Correctness proof
  - Show that M runs in polynomial time — Time complexity analysis

design a Turing machine $\equiv$ design an algorithm

Use Polynomial Time Verifier to Prove that A is in **NP**

- Definition: A ***verifier*** for a language A is an algorithm V, where

$$A = \{w \mid V \text{ accepts } \langle w, c\rangle \text{ for some string } c\}.$$

Prove A is in **NP** ⇔ Design a **polynomial time verifier** to decide A (with help from some c)

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<Proof Idea>

1. Assume that there is a certificate c with size polynomial in the length of w
2. Design a verifier V on input $\langle w, c \rangle$ that *accepts* all $w \in A$ and *rejects* all $w \notin A$
3. Show that V runs in polynomial time (in the length of w)

CLIQUE is in **NP**

- CLIQUE $= \{\langle G, k\rangle \mid G$ is an undirected graph with a k-clique$\}$
  - A string c that encodes a clique with size k in G is a good certificate

Prove CLIQUE is in **NP** ⇔ Design a **polynomial time verifier** to decide CLIQUE (with help from some c)

<Proof Idea>

1. Assume that there is a certificate c that encodes a clique with size k in G.
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Let string c that encodes a clique with size k in G as a certificate

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Prove CLIQUE is in **NP** ⇔ Design a polynomial time verifier to decide CLIQUE (with help from some c)

<Proof>

Let string c that encodes a clique with size k in G as a certificate

V = "On input $\langle\langle G, k\rangle, c\rangle$:

1. Test whether c is a set of k nodes in G
2. Test whether G contains all edges connecting nodes in c
3. If both 1 and 2 pass, *accept*; otherwise, *reject*."

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<Proof Idea>

1. Show that for any yes instance w, there is a polynomial-size certificate c.
2. Design a verifier V on input $\langle w, c \rangle$ that *accepts* all $w \in A$ and *rejects* all $w \notin A$
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SUBSET-SUM is in **NP**

- SUBSET-SUM $= \{\langle S, t\rangle \mid S = \{x_1, \cdots, x_k\}$ and for some $\{y_1, \cdots, y_m\} \subseteq \{x_1, \cdots, x_k\}$, we have $\Sigma y_i = t\}$
- Prove SUBSET-SUM is in **NP** $\Leftrightarrow$ Design a **polynomial time** **verifier** to decide CLIQUE (with help from some c)

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1. Test whether $|c| < |S|$
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Step 1 takes at most linear time to scan through the input. Step 2 takes $|c| < |S|$ summations. Step 3 takes at most $|c|$ times of scanning through the input. Hence, V runs in polynomial time in the input length.

Boolean Formula

- Boolean formula: an expression involving Boolean variables and operations
  - Example:
    - *literals*
    - $\phi = \bar{x} \wedge y \wedge z$
    - $\phi = (\bar{x} \wedge y) \vee (x \wedge \bar{z})$
  - (Boolean) variables: x, y, z
- The Boolean variables can take on the values *TRUE* (1) and *FALSE* (0)

Boolean Formula

- Boolean formula: an expression involving Boolean variables and operations
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    - $\phi = \underset{0}{\bar{x}} \wedge \underset{1}{y} \wedge \underset{1}{z}$
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Boolean Formula

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    - $\phi = \underset{0}{\bar{x}} \wedge \underset{1}{y} \wedge \underset{1}{z}$ = FALSE $\quad x =$ TRUE, $y =$ TRUE, $z =$ TRUE
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- A Boolean formula is **satisfiable** if some assignment of *TRUE*s and *FALSE*s to the variables make the formula true
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SAT is in **NP**

- SAT $= \{\langle \phi \rangle \mid \phi$ is a satisfiable Boolean formula$\}$
- Prove SAT is in **NP** ⇔ Design a polynomial time verifier to decide SAT (with help from some c)

<Proof>

Let string c that encodes a truth assignment of variables in ϕ as a certificate

V = "On input $\langle\langle \phi \rangle, c\rangle$:

1. Replace the literals in ϕ by the truth assignments in c
2. Test whether the resulting ϕ is true
3. If 2 pass, *accept*; otherwise, *reject*."

For each replacement in Step 1, it takes at most linear time of scanning through the input. In total, it scan through the input ℓ times, where ℓ is the number of literals in ϕ. Step 2 can be done in one scan through the input. Hence, V runs in polynomial time in the input length.

D-HAM-PATH

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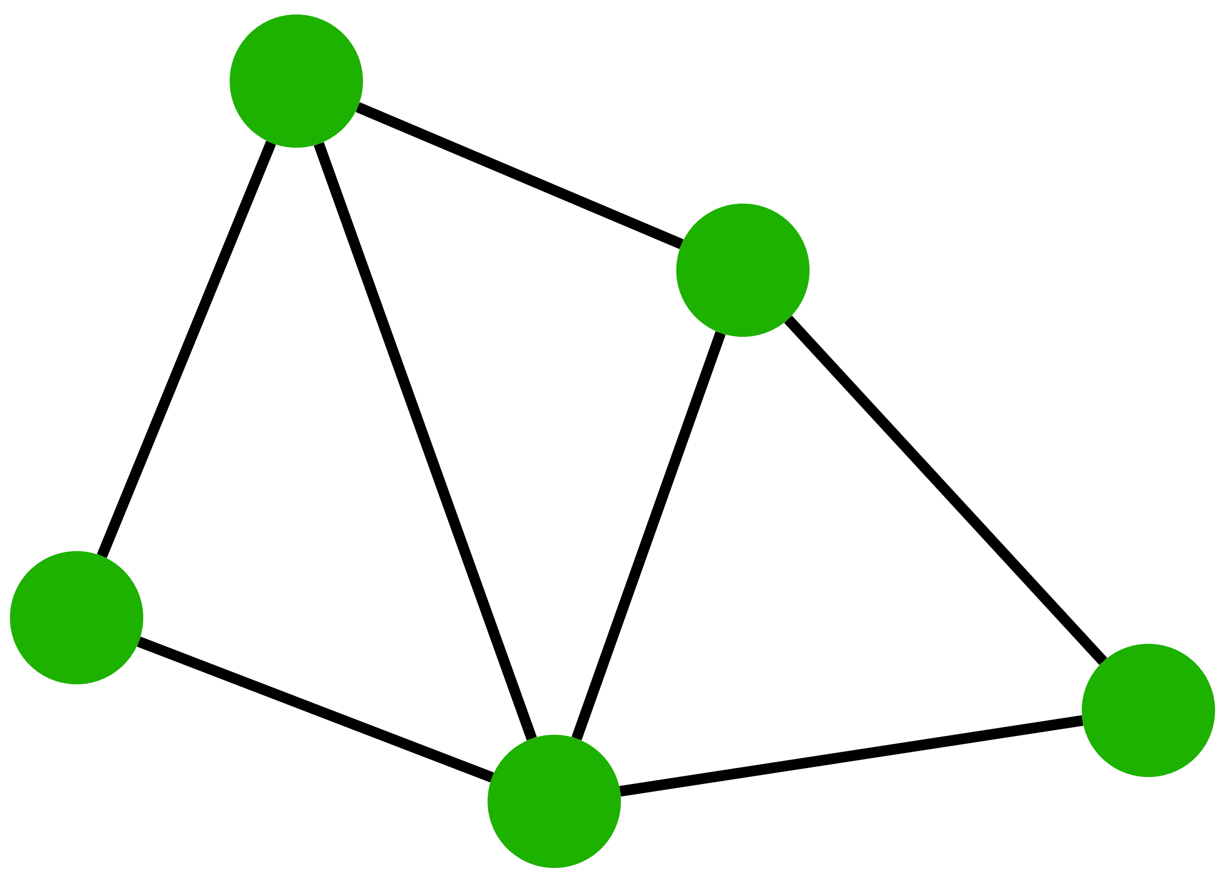
- A **Hamiltonian path** of a graph $G = (V, E)$ is a simple path that contains each vertex in V.



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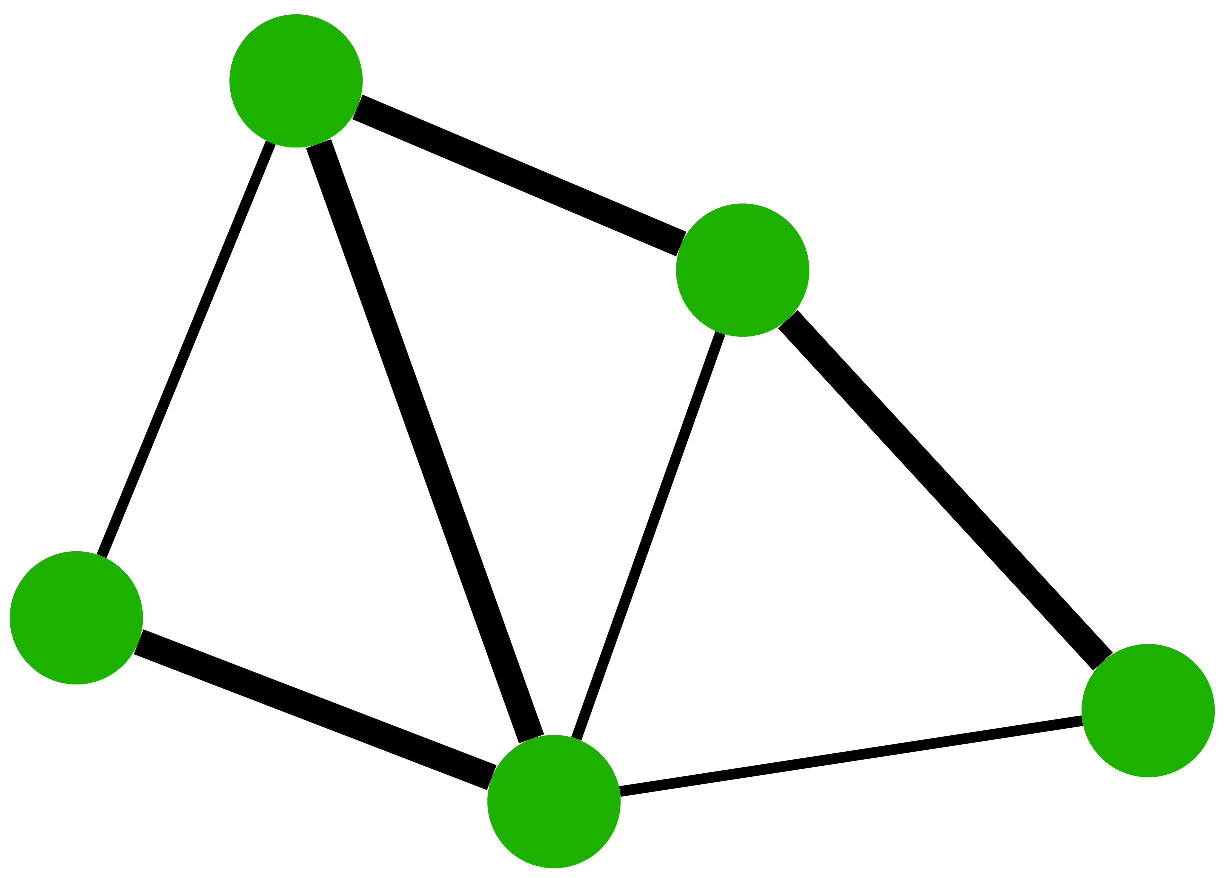

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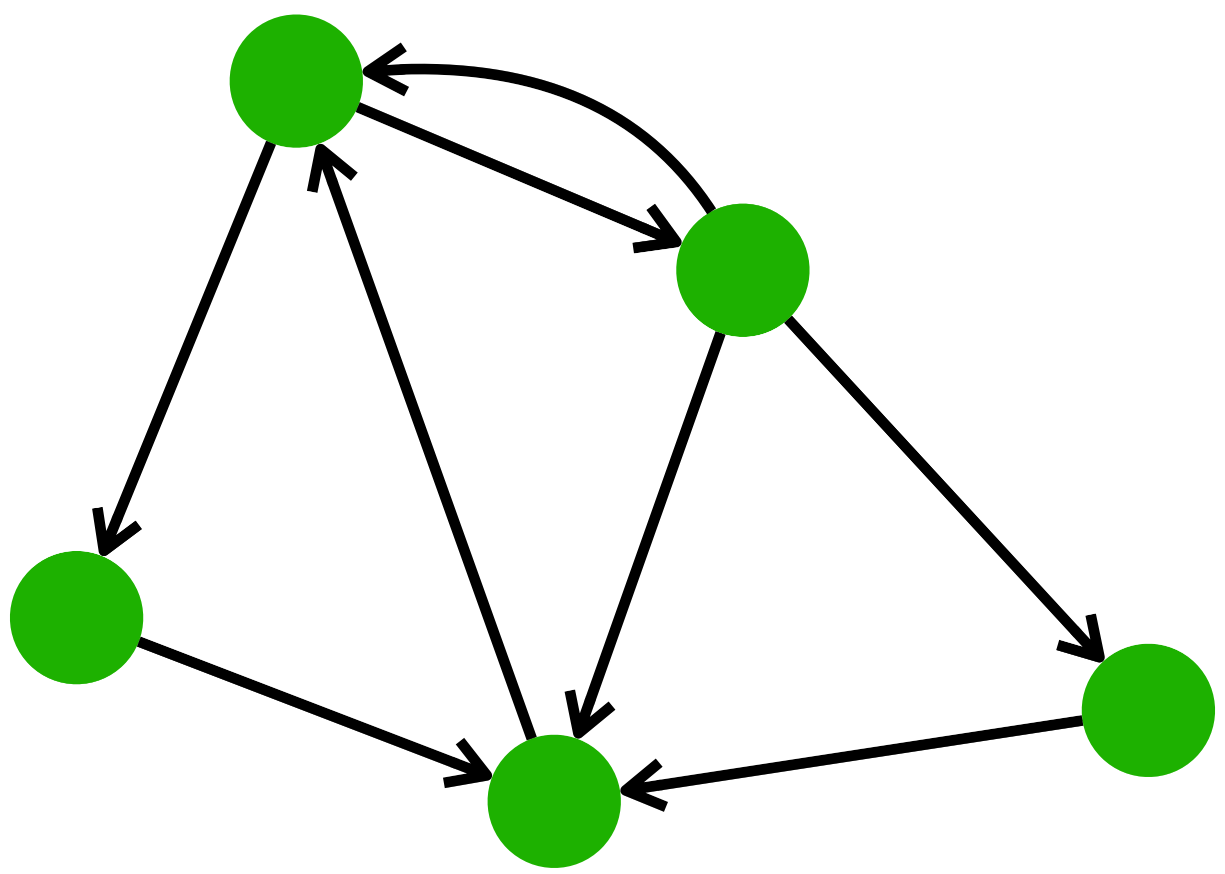

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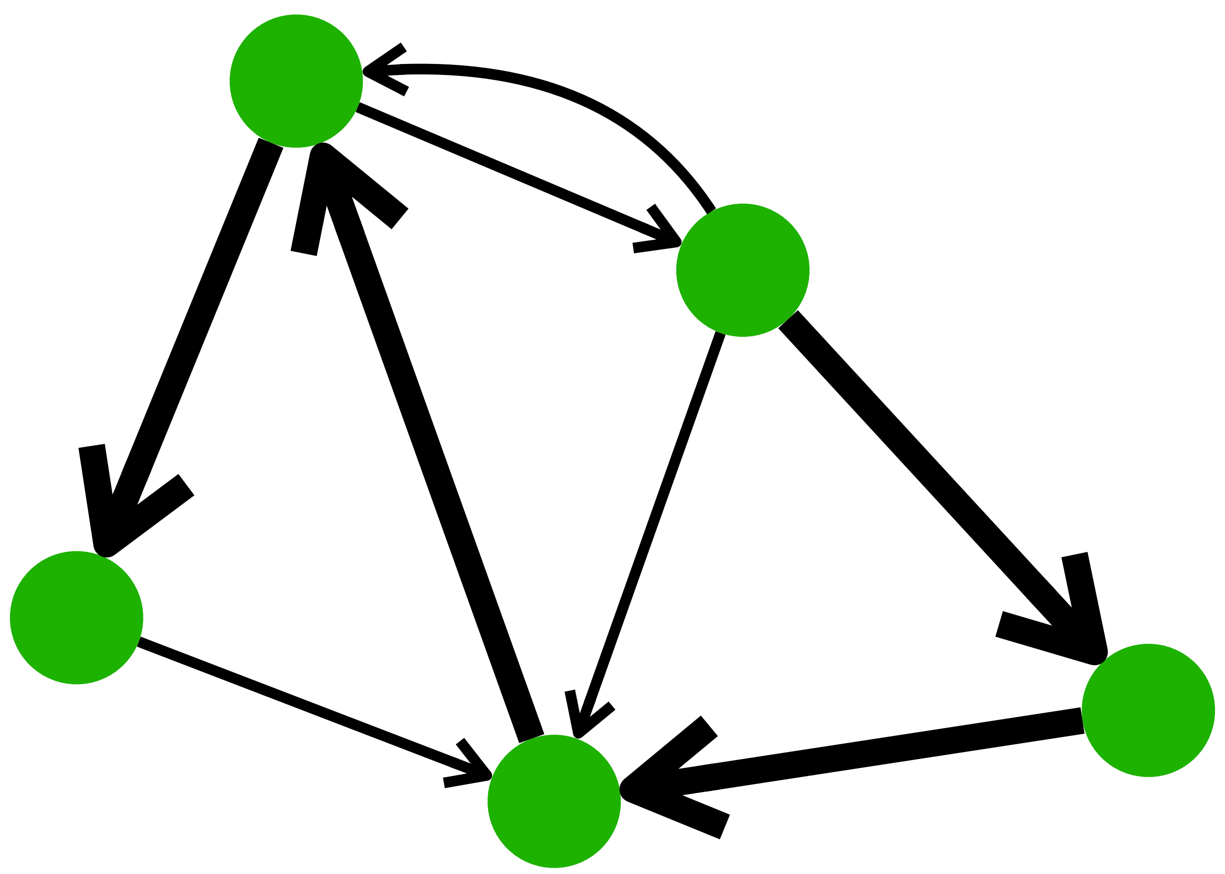
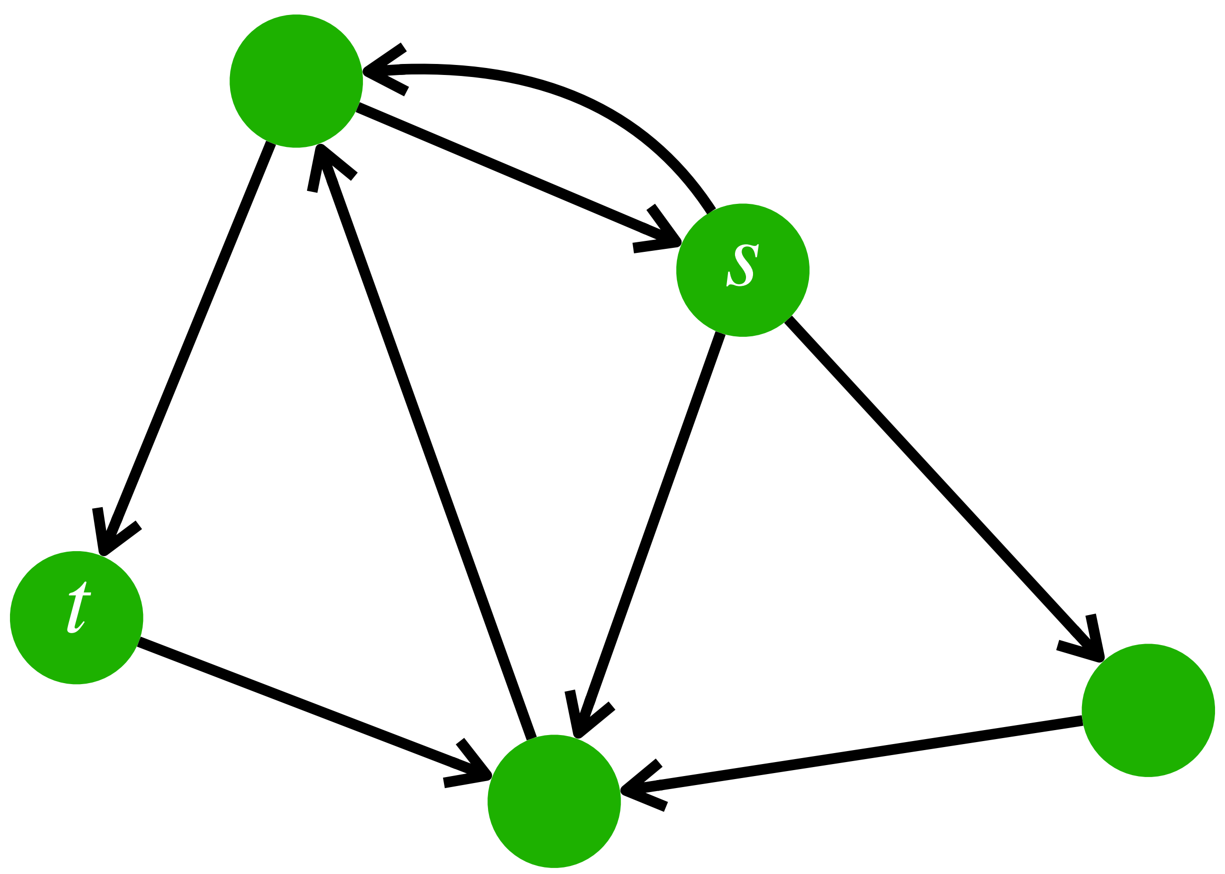


D-HAM-PATH

- A **Hamiltonian path** of a graph $G = (V, E)$ is a simple path that contains each vertex in V.

- D-HAM-PATH $= \{\langle G, s, t\rangle |\ G$ is a directed graph with a Hamiltonian path from s to $t\}$



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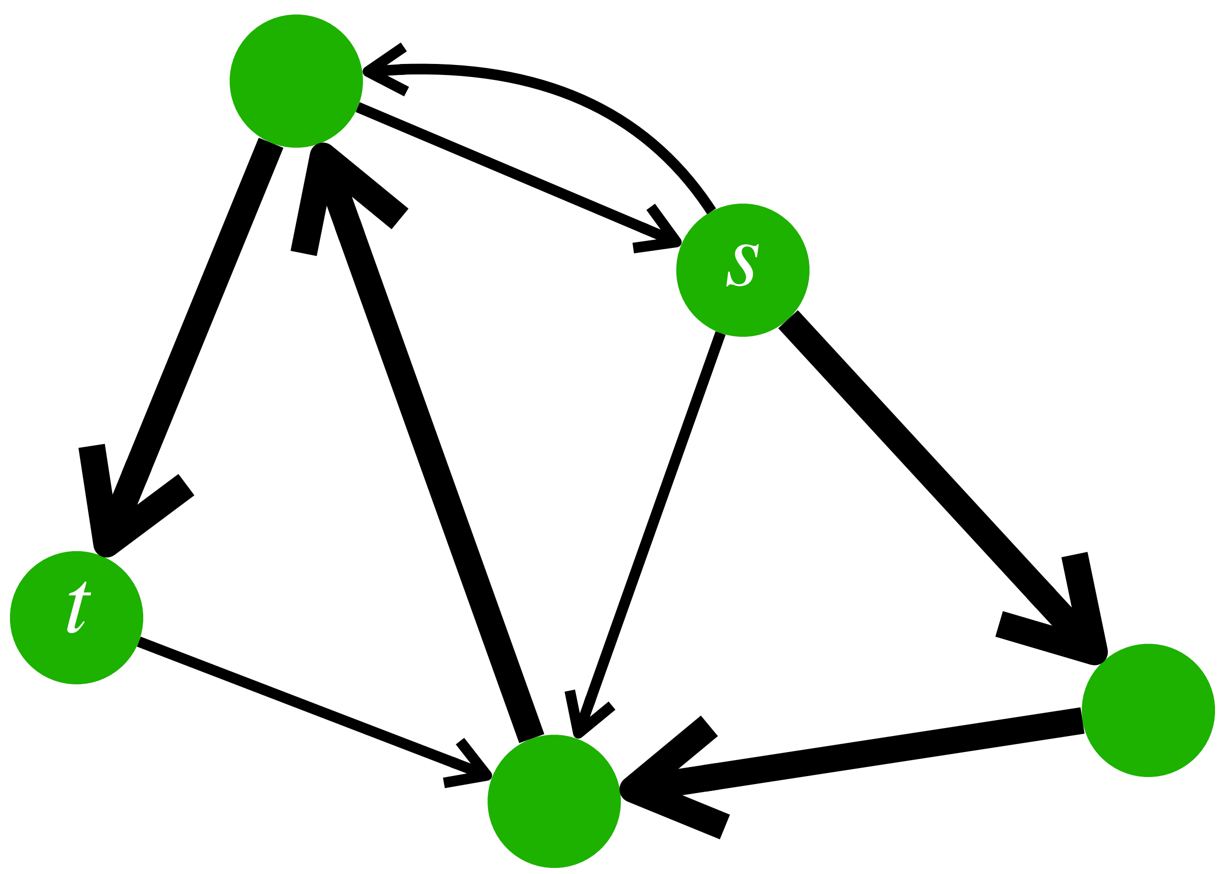

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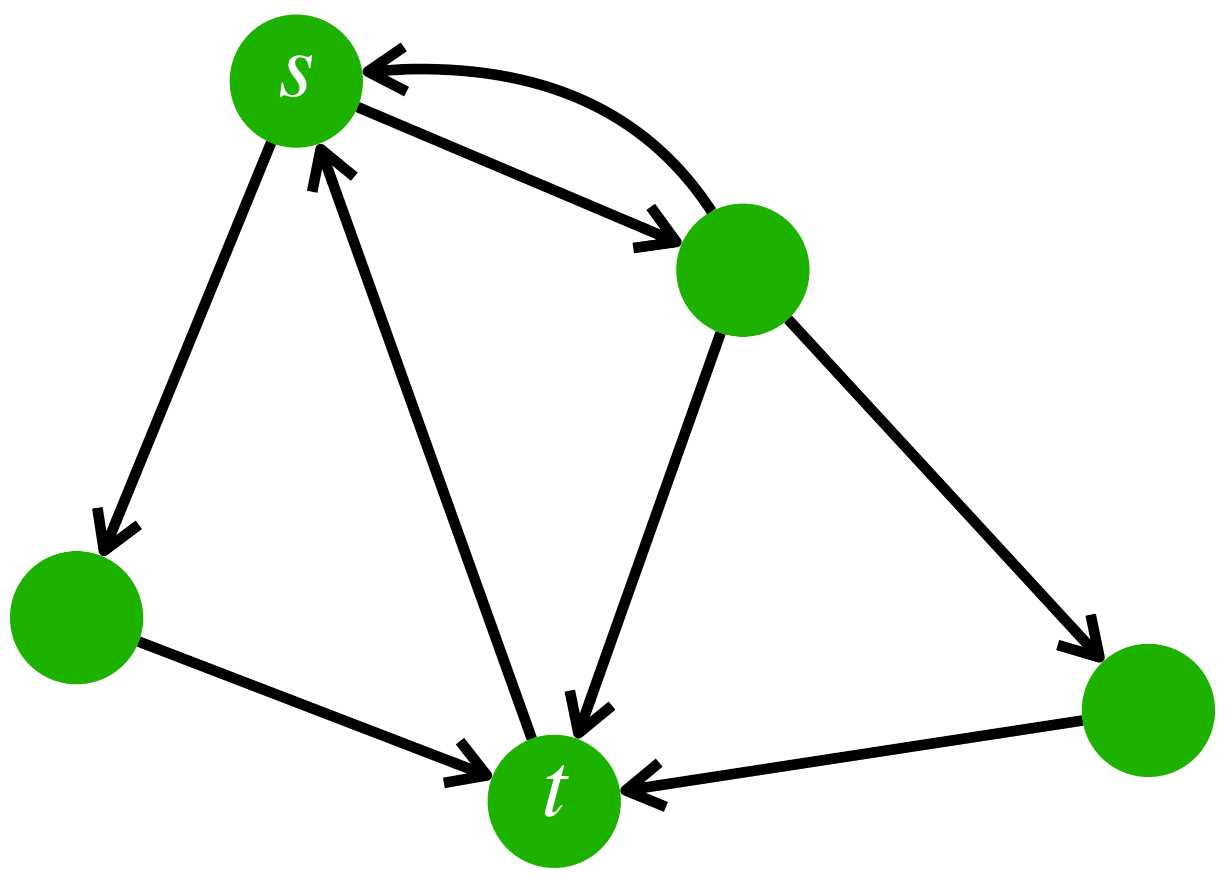


D-HAM-PATH is in **NP**

- D-HAM-PATH $= \{\langle G, s, t\rangle | \, G$ is a directed graph with a Hamiltonian path from s to $t\}$
- Prove D-HAM-PATH is in **NP** ⇔ Design a polynomial time verifier to decide D-HAM-PATH (with help from some c)

<Proof>

Let string c that encodes a permutation of vertices in G that forms a Hamiltonian walk starting from s and end at t as a certificate

V = "On input $\langle\langle G, s, t\rangle, c\rangle$:

1. Check if c is indeed a permutation of vertices in G starting from s and end at t
2. For every consecutive pair of vertices in c, v_i and v_{i+1}, check if there is an edge from v_i to v_{i+1} in G
3. If 1 and 2 both pass, *accept*; otherwise, *reject*."

For each element in c in Step 1, it takes at most linear time of scanning through the input. In total, it scan through the input n times, where n is the number of vertices in G. Each consecutive pair in Step 2 can be checked in one scan through the input, and there are at most $O(n)$ pairs. Hence, V runs in polynomial time in the input length.

What Happened

- To show that a problem is in **NP**, we can show that it is polynomial-time verifiable

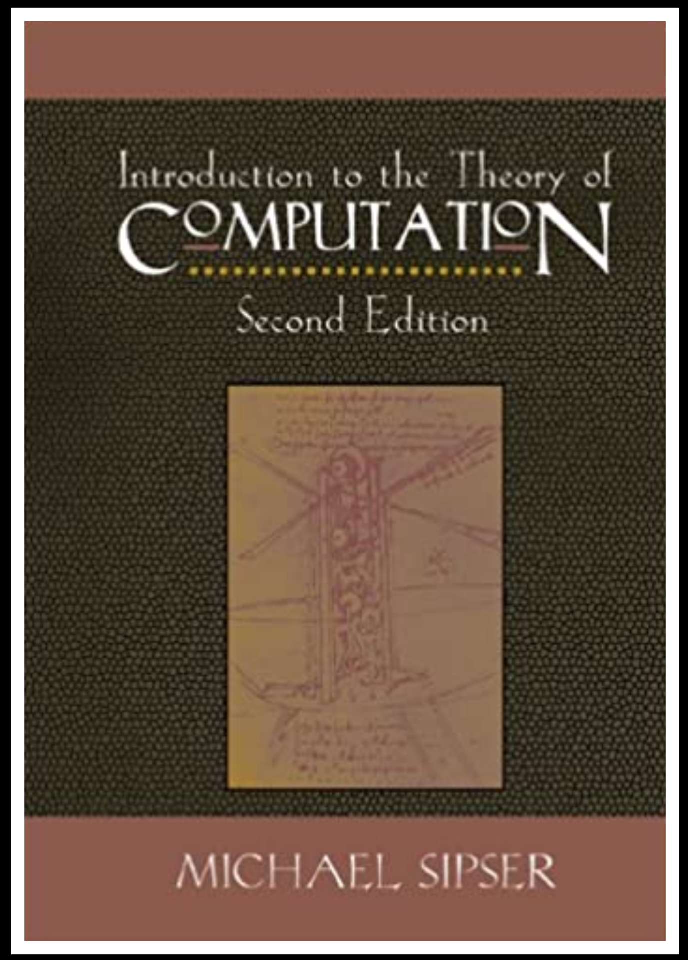
<Proof Idea>

1. Show that for any yes instance w, there is a polynomial-size certificate c.
2. Design a verifier V on input $\langle w, c \rangle$ that *accepts* all $w \in A$ and *rejects* all $w \notin A$
3. Show that V runs in polynomial time (in the length of w)

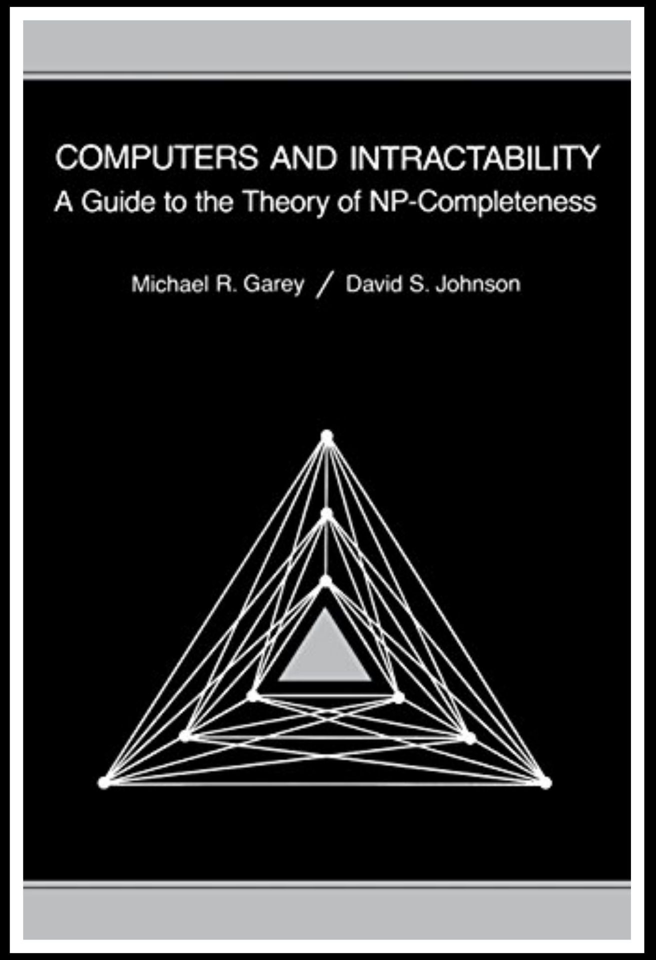
Why **P** and **NP**?

- There are many $f(n)$ time Turing machine variations that have an equivalent poly($f(n)$) time single-tape Turing machine
- Every $f(n)$ time non-deterministic Turing machine has an equivalent $2^{O(f(n))}$ time deterministic Turing machine
- There is at most a square or polynomial difference between the time complexity of problems measured on deterministic single-tape and many Turing machine variations
- There is at most an exponential difference between the time complexity of problems on deterministic and nondeterministic Turing machines

Reference



- *Introduction to the Theory of Computation* by Michael Sipser



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- *Introduction to Algorithms* by Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein

- *Computational Complexity* by Christos h. Papadimitriou