Algorithms for Decision Support

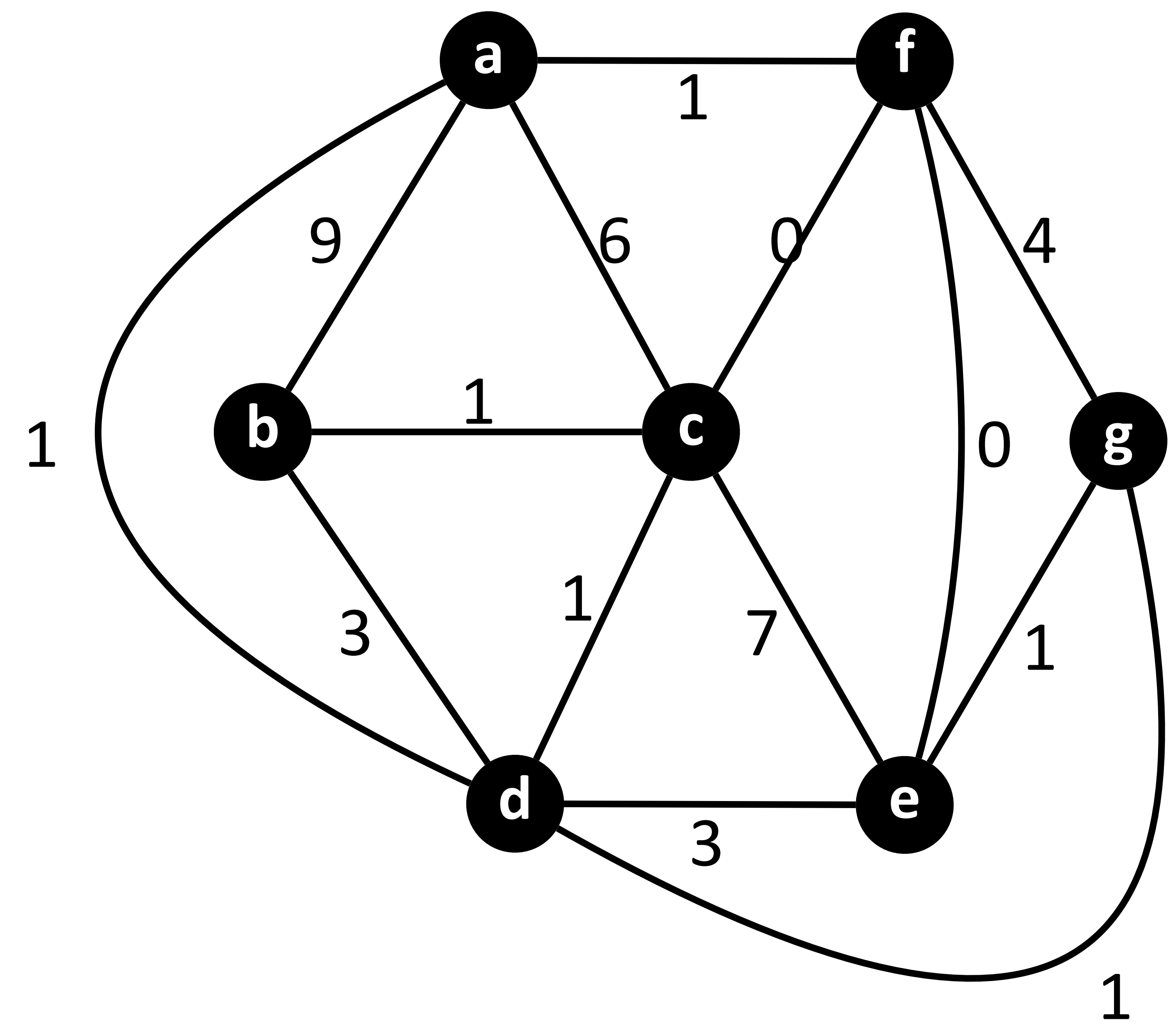
(Integer) Linear Programming (3/3)

Outline

- **Warm up: Minimum spanning tree**
- Tricks:
  - Range constraints
  - Absolute value objective
  - Min-max objective
  - Discontinuous-values variables
  - Fixed-cost objective
    - Facility location
    - Lot-sizing
  - Or and conditional conditions
- Solving ILP: Cutting plane

Minimum Spanning Tree

- Given a graph $G = (V, E)$ and edge weights c_{uv} for $(u, v) \in E$, find a minimum weight subgraph such that the subgraph is connected.



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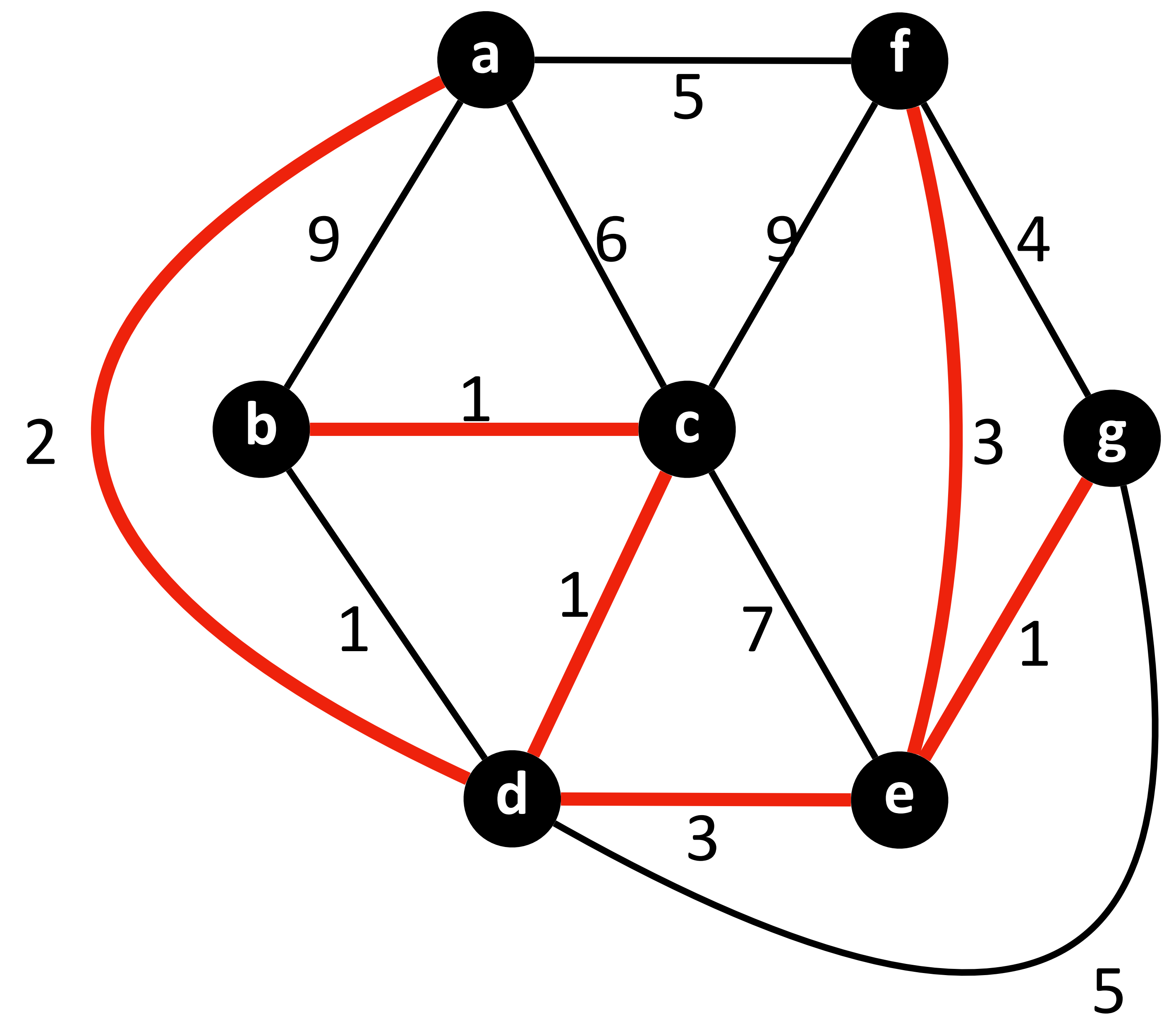

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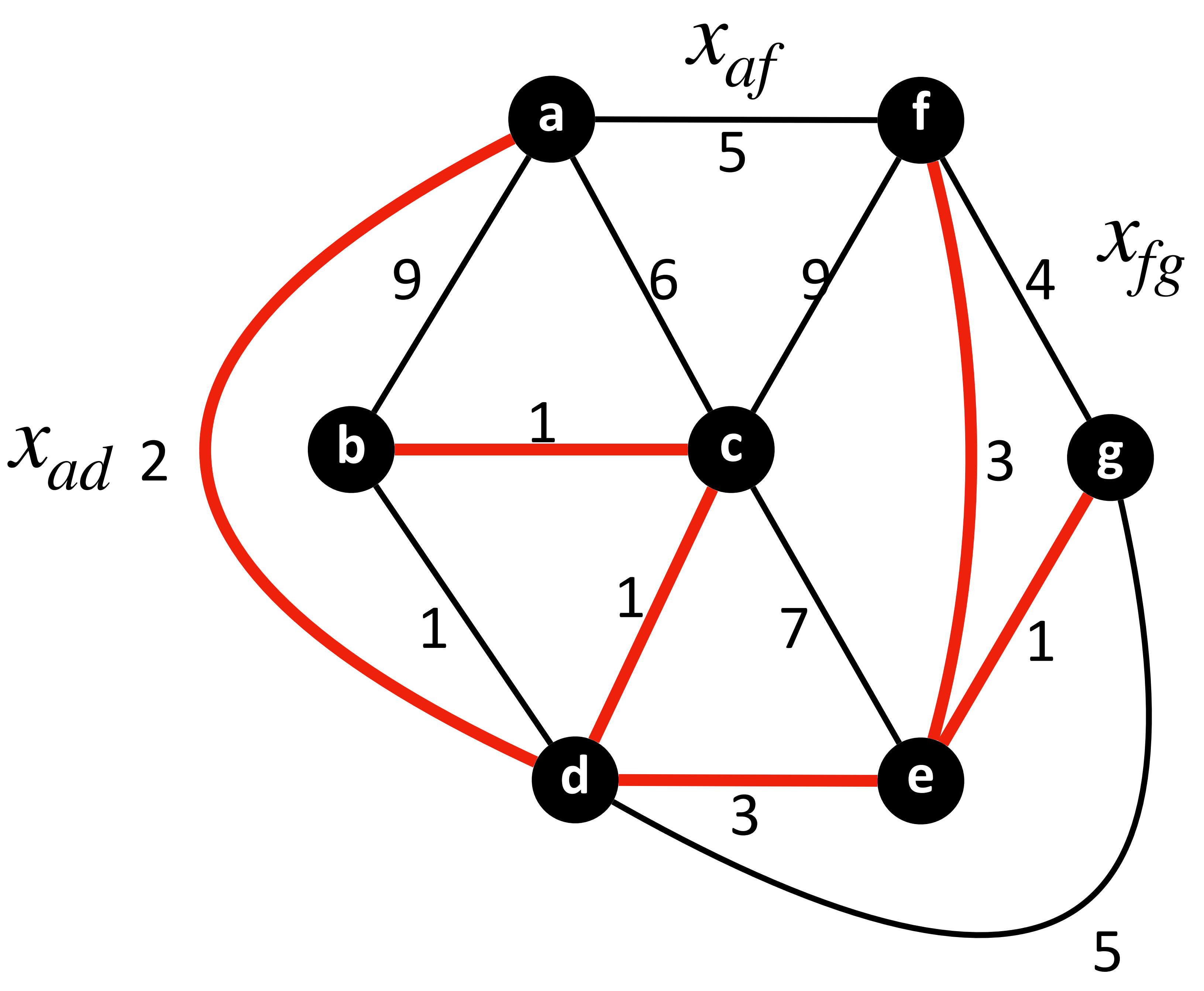

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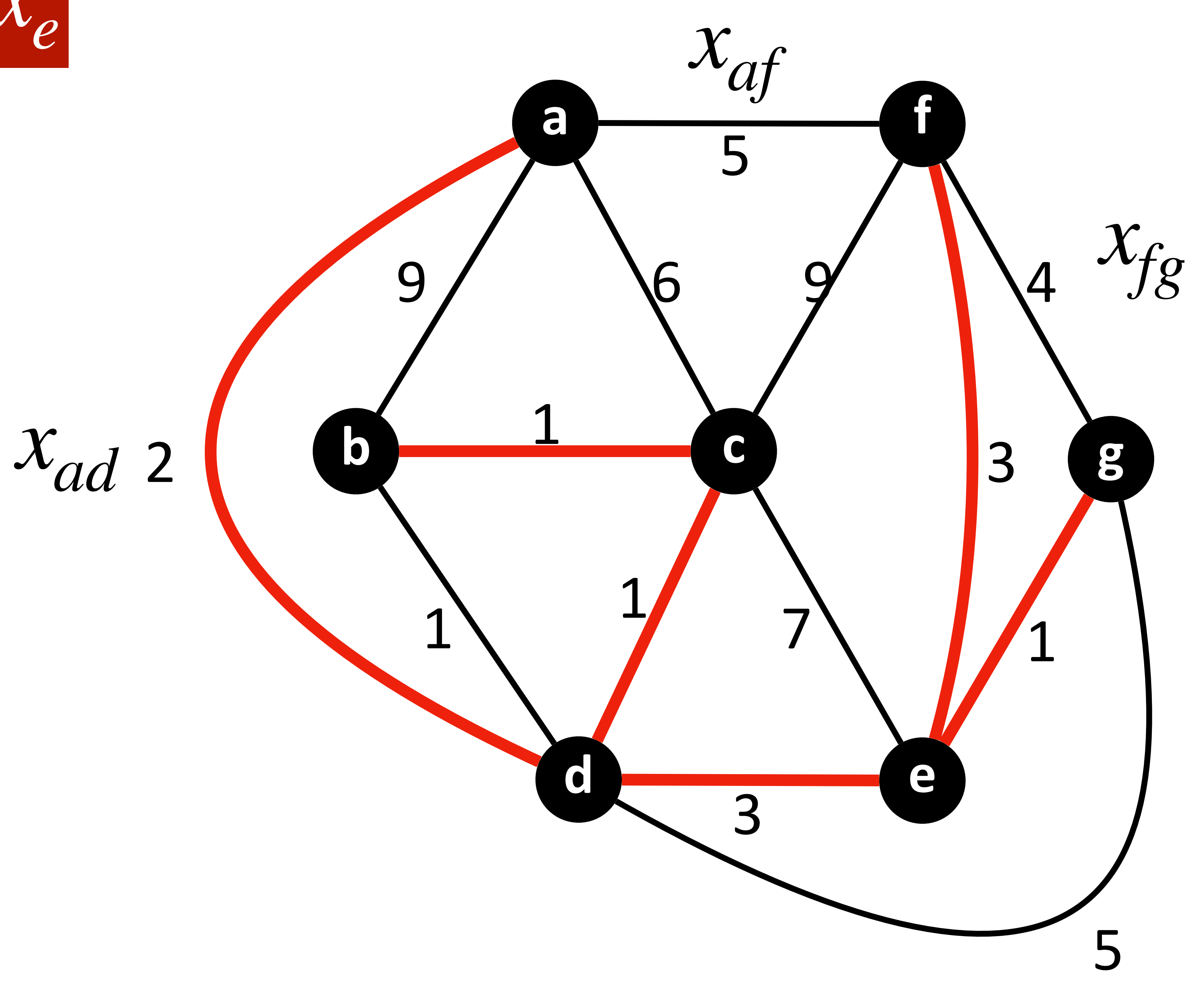
$\min \Sigma_{e \in E} c_e x_e$



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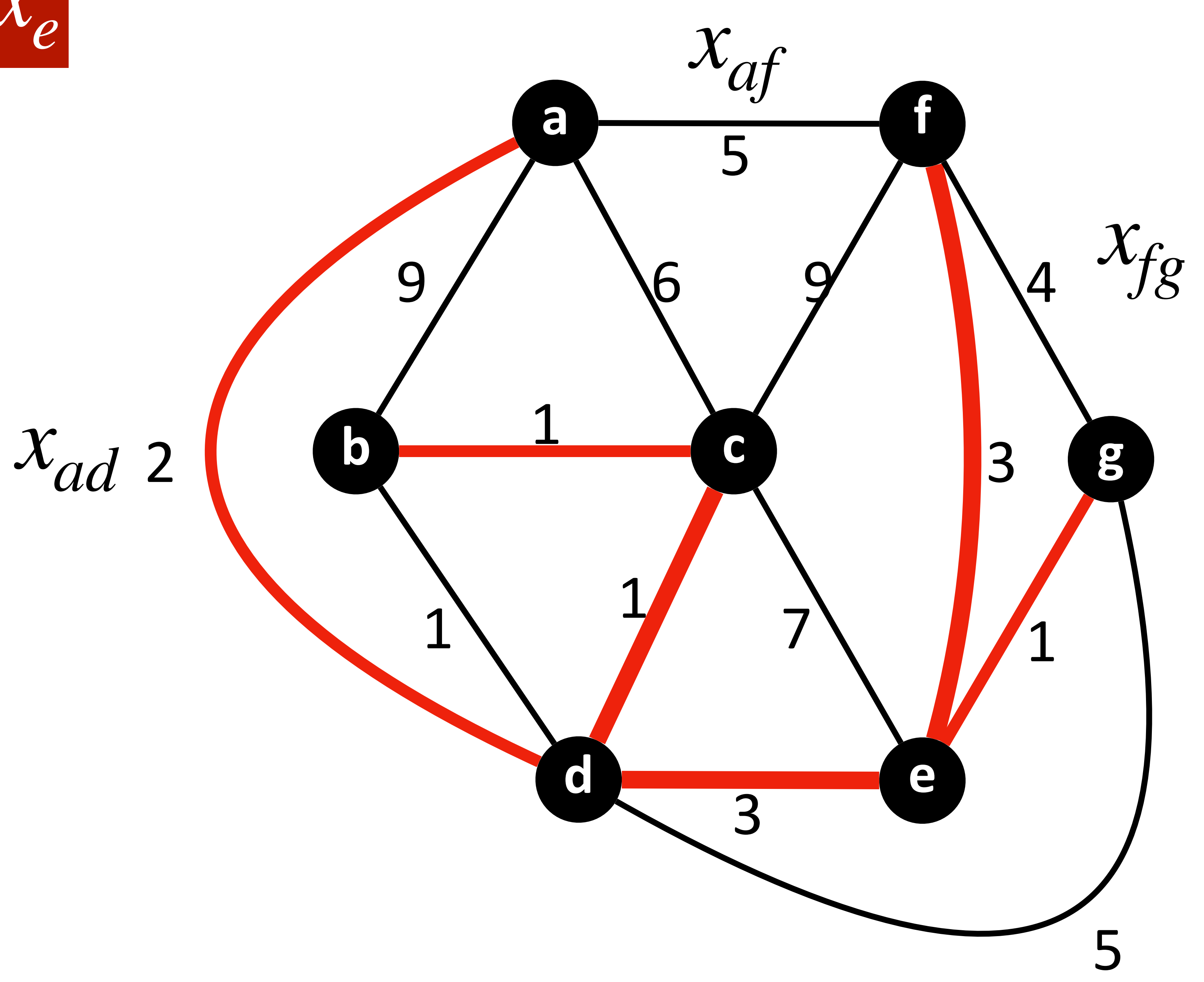
There is a path between any two vertices



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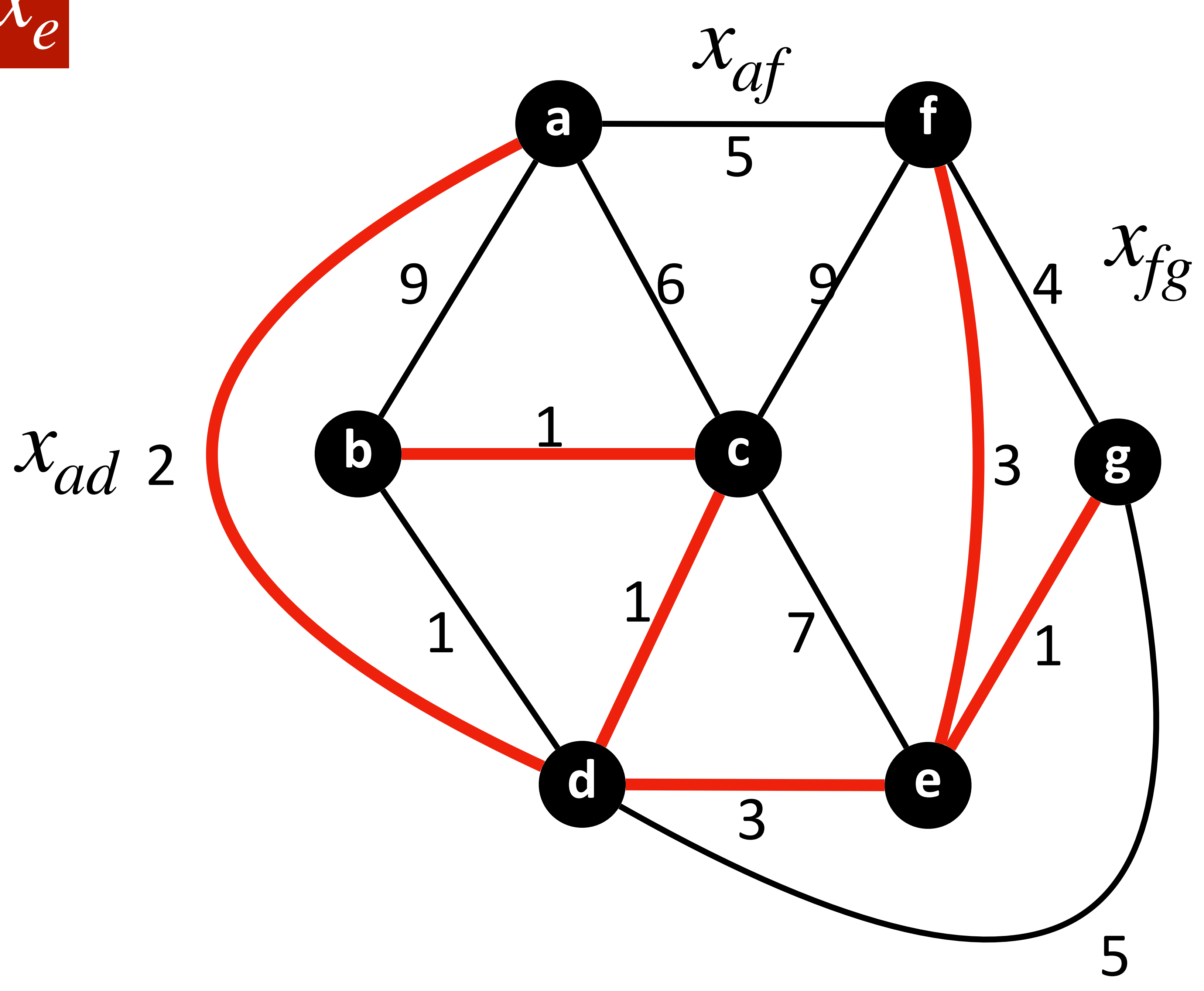

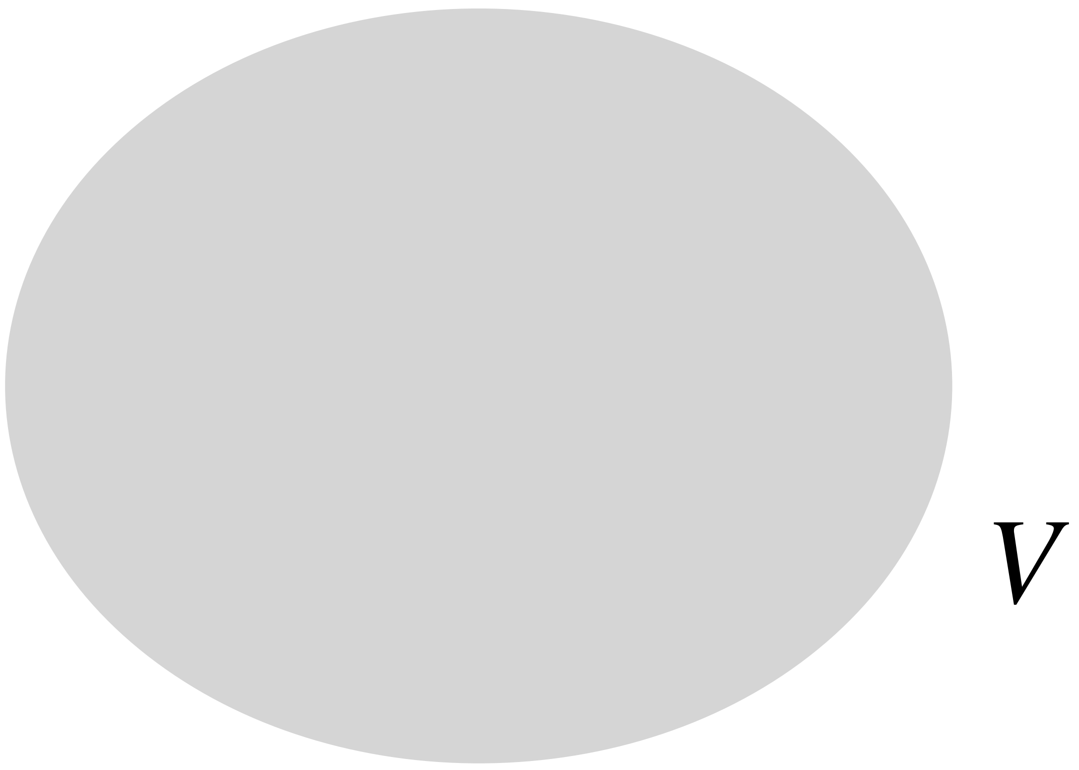

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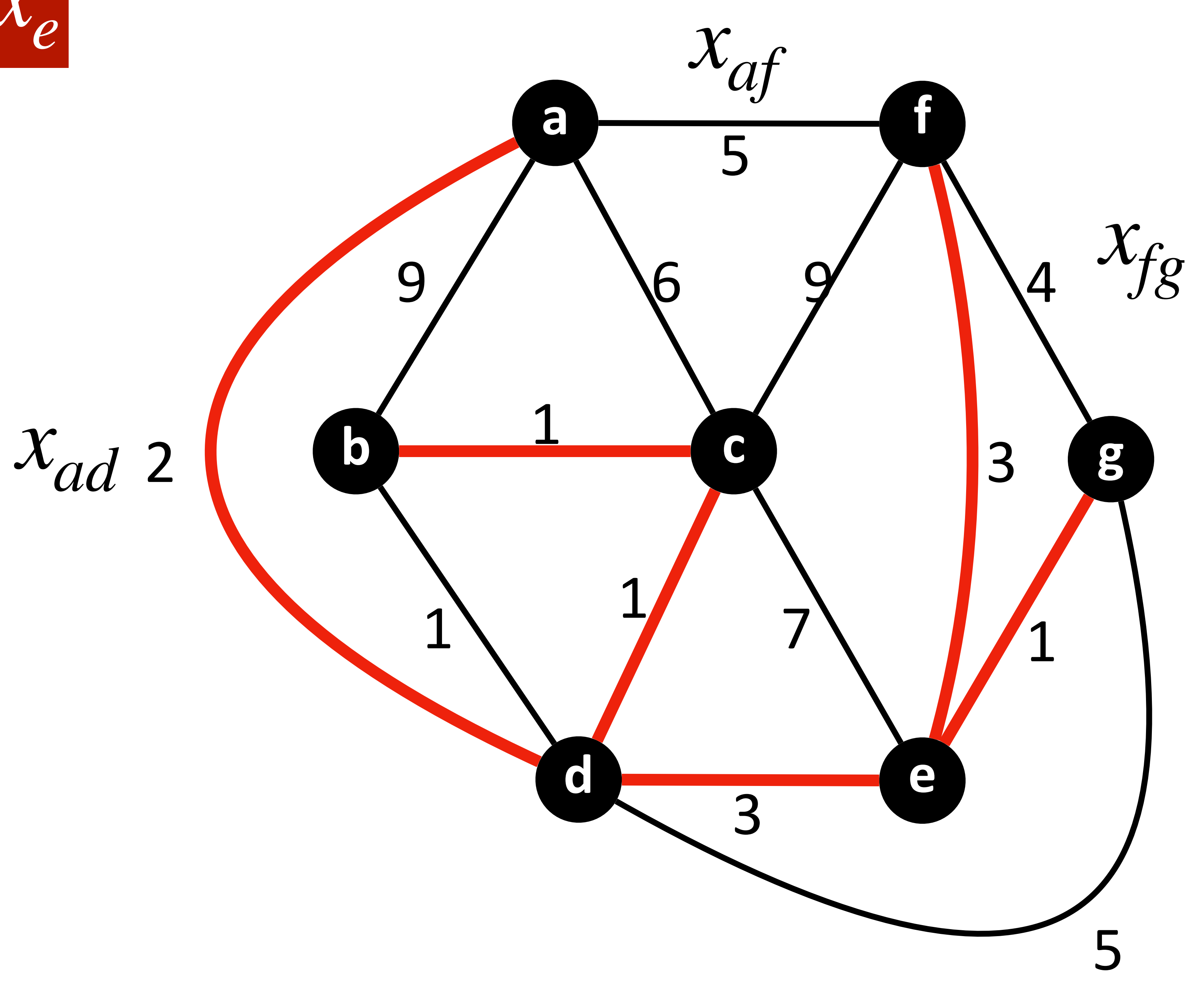

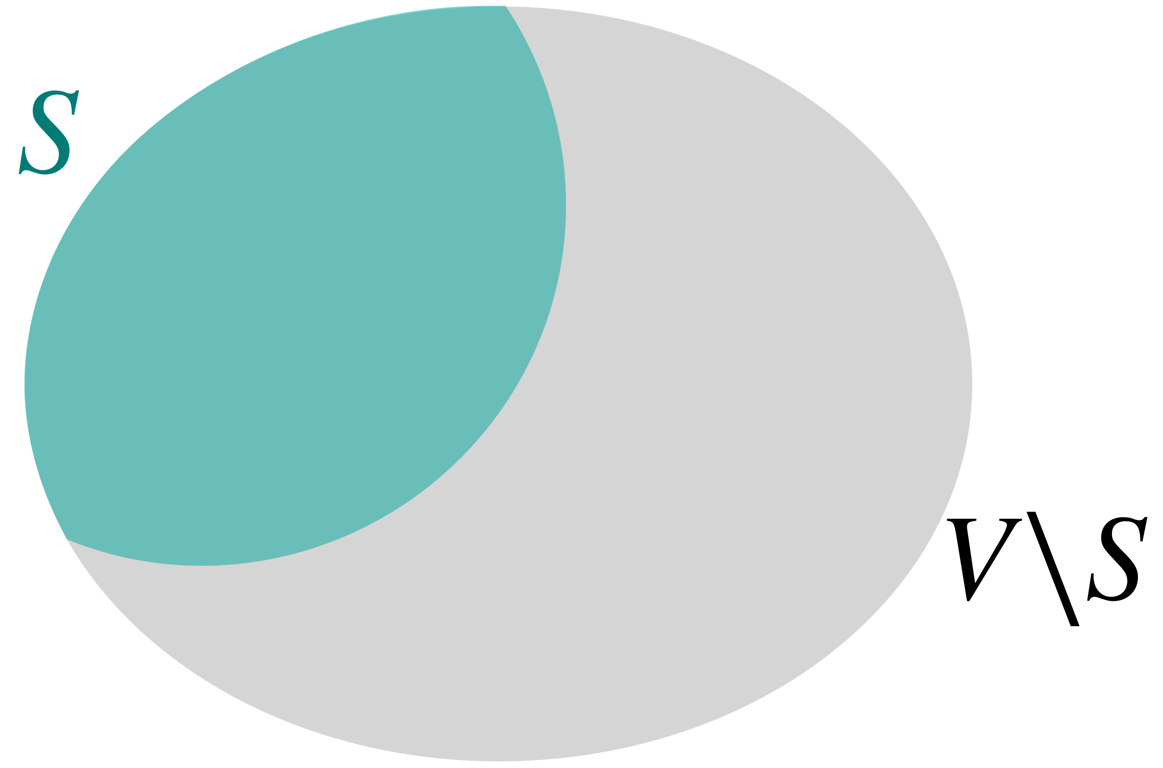

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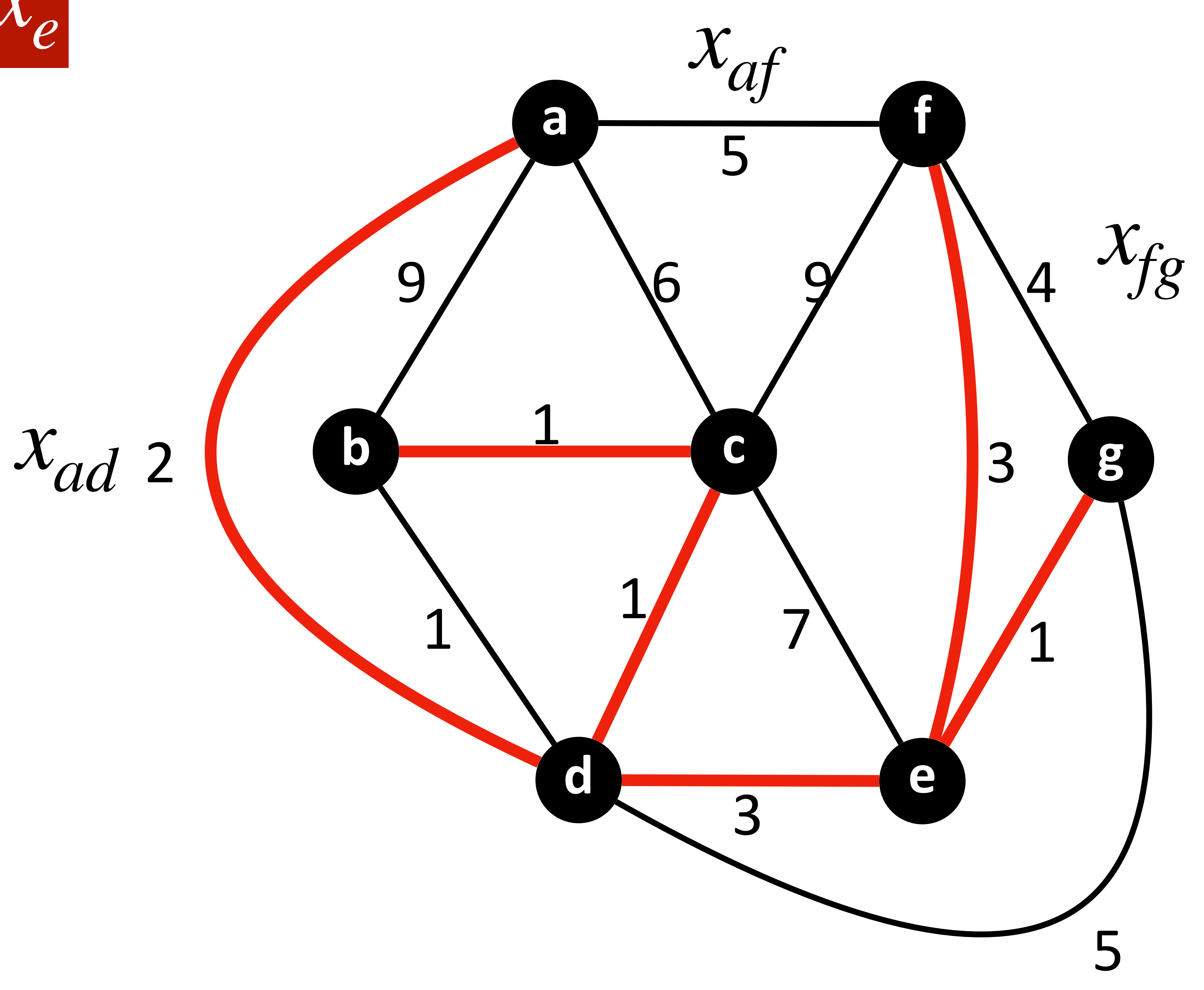

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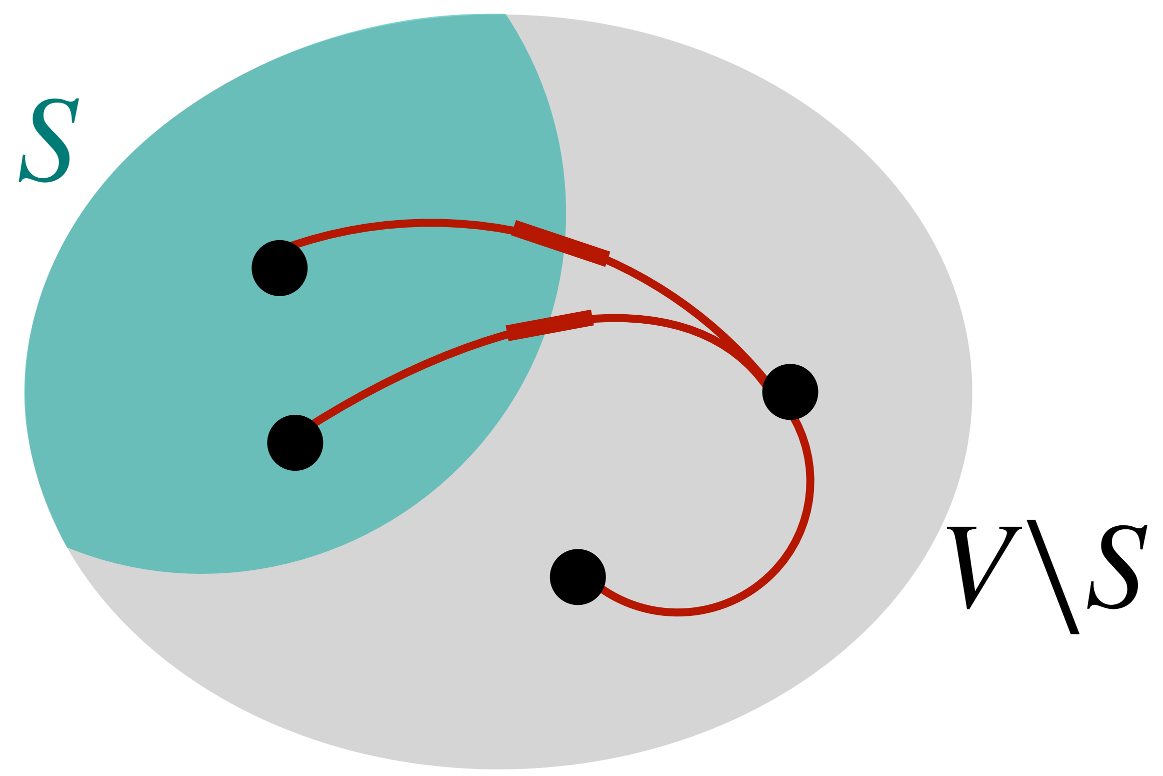

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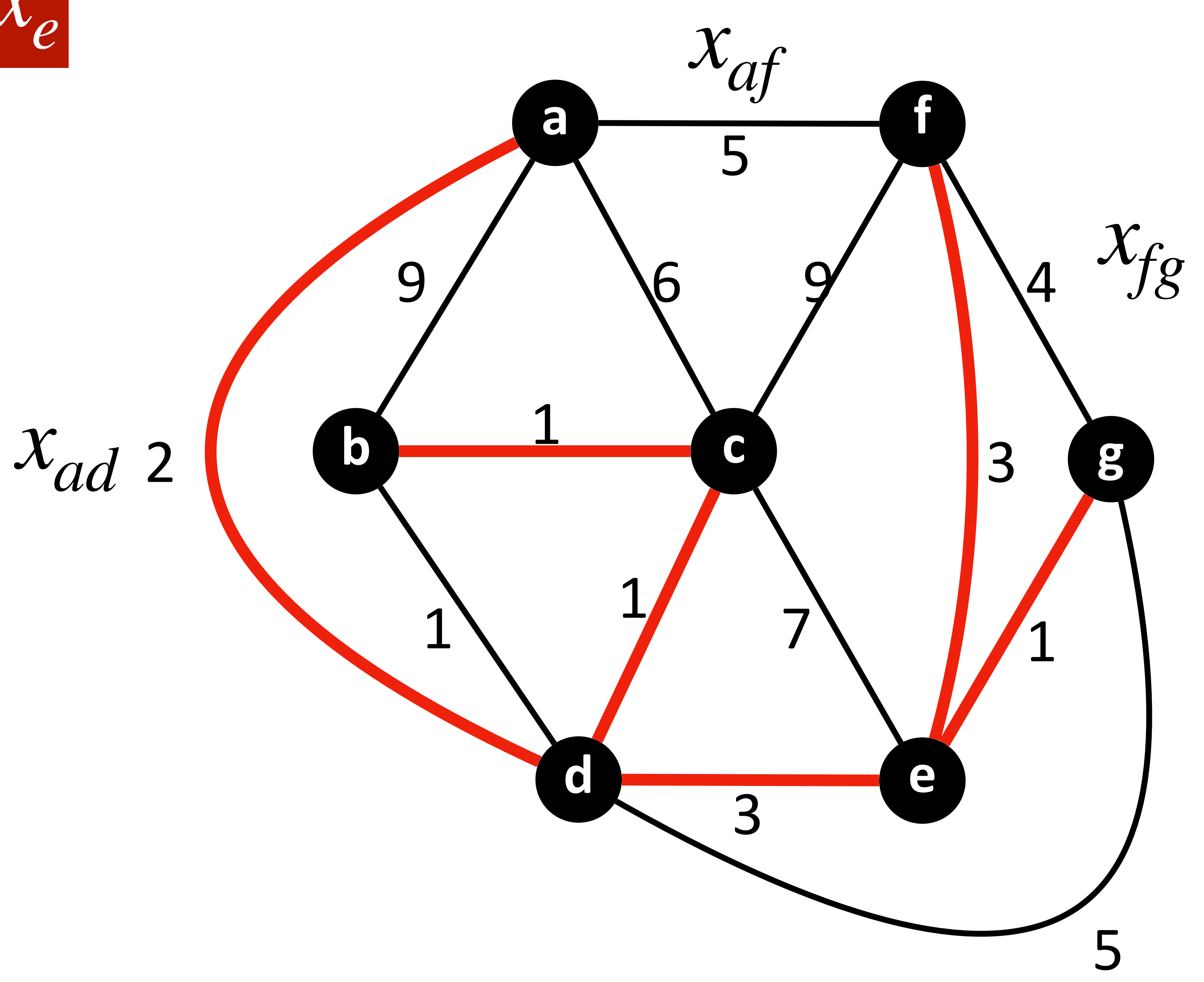

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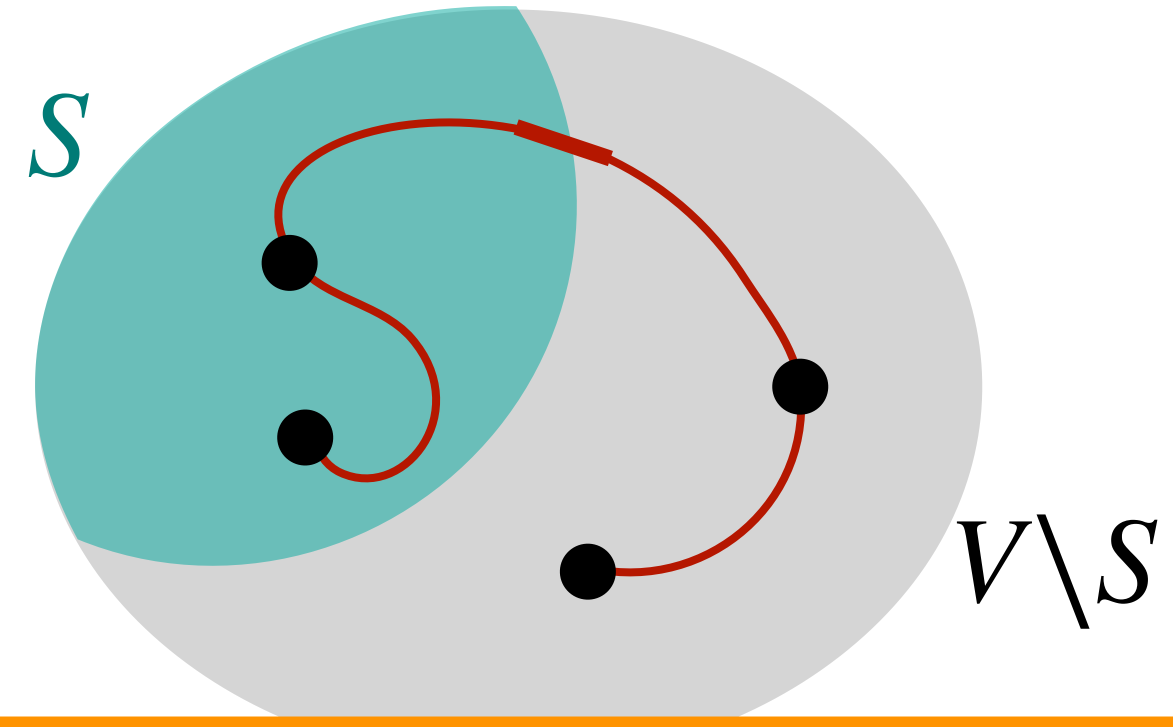

$\Sigma_{(u,v):u \in S, v \in V \backslash S}\, x_{uv} \geq 1$ for all $S \subset V$ and $1 \leq |S| < n$

Minimum Spanning Tree

- Variables: $x_{uv} = 1$ if the edge (u, v) is in the subgraph, and $x_{uv} = 0$ otherwise

- minimize $\quad \Sigma_{(u,v)\in E}\, c_{uv}x_{uv}$

subject to $\Sigma_{(u,v):u\in S,v\in V\setminus S}\, x_{uv} \geq 1$ for any subset $S \subset V$ with $1 \leq |S| < n$

$x_{uv} \in \{0,1\}$ for $(u, v) \in E$

Tips

- Observe the problem itself to get the constraints

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Range constraints 1

LP1

Minimize $\sum_{j \in J} c_j x_j$

s. t. $\ell_i \leq \sum_{j \in J} a_{ij} x_j \leq u_i$ for all i

$x_j \geq 0$

LP2

Minimize $\sum_{j \in J} c_j x_j$

s. t. $\sum_{j \in J} a_{ij} x_j \leq u_i$ for all i

$-\sum_{j \in J} a_{ij} x_j \leq -\ell_i$ for all i

$x_j \geq 0$ for all j

Range constraints 2

LP1

Minimize $\sum_{j \in J} c_j x_j$

s. t. $\ell_i \leq \sum_{j \in J} a_{ij} x_j \leq u_i$ for all i

$x_j \geq 0$

LP2'

Minimize $\sum_{j \in J} c_j x_j$

s. t. $d_i + \sum_{j \in J} a_{ij} x_j = u_i$ for all i

$d_i \leq u_i - \ell_i$ for all i

$x_j \geq 0$ for all j

$d_i \geq 0$

Range constraints 3

LP1

Minimize $\sum_{j \in J} c_j x_j$

s. t. $\sum_{j \in J} a_{ij} x_j = b_i$ for all i

$x_j \geq 0$

LP2'

Minimize $\sum_{j \in J} c_j x_j$

s. t. $\sum_{j \in J} a_{ij} x_j \leq b_i$ for all i

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$\sum_{j \in J} a_{ij} x_j \geq b_i$ for all i

$x_j \geq 0$ for all j

Tips

- Replacing the equality constraints by $\geq$-constraint and $\leq$-constraint

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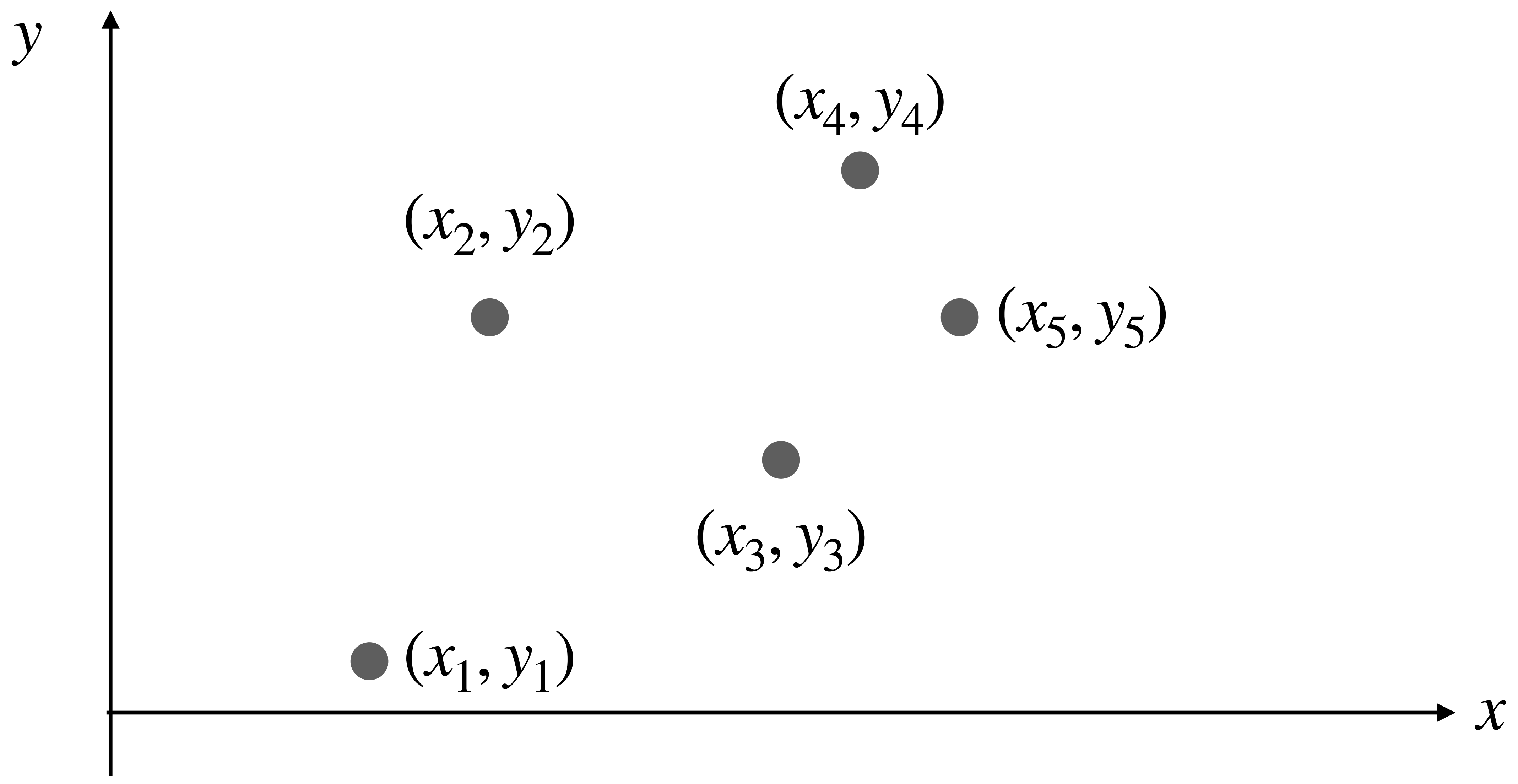
Least absolute deviations estimation



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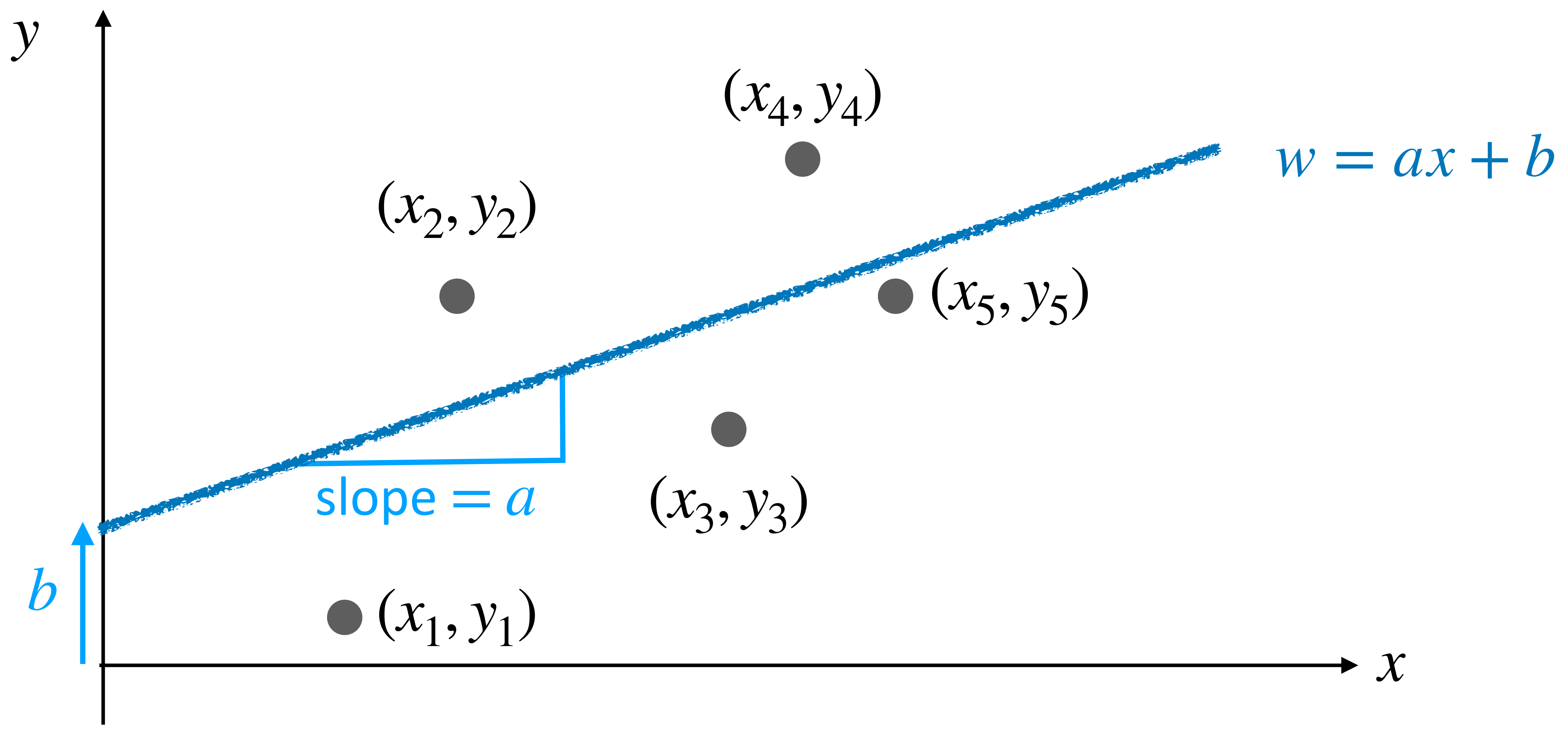

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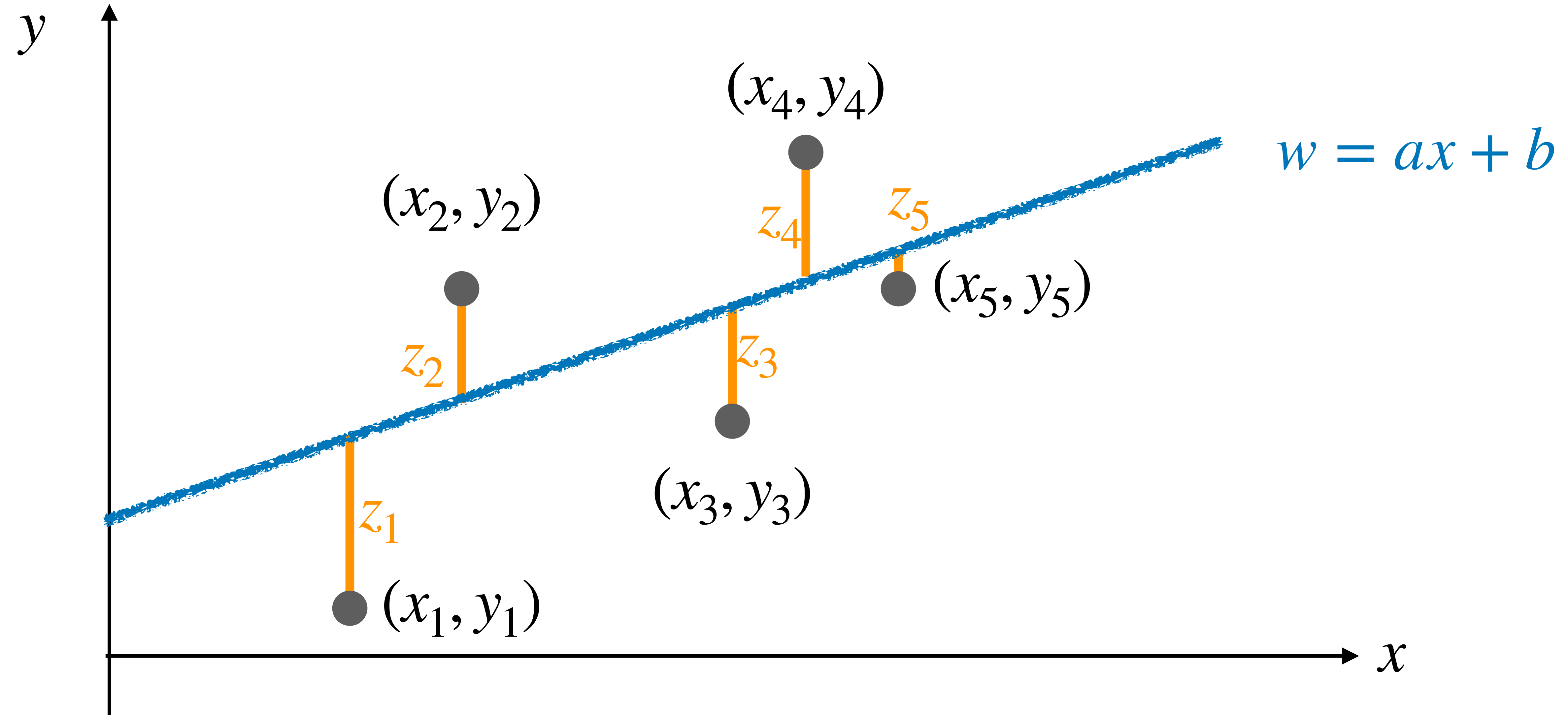

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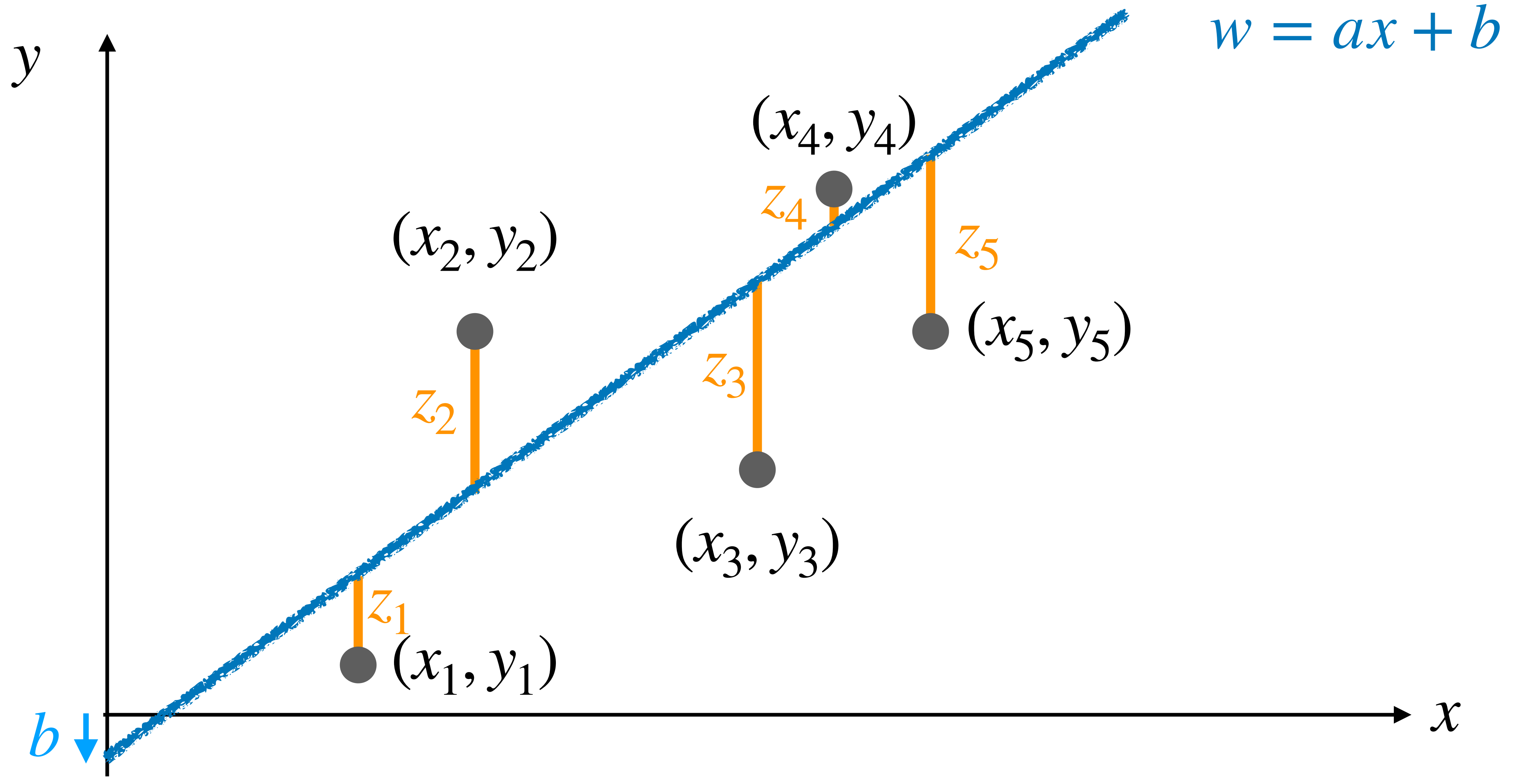

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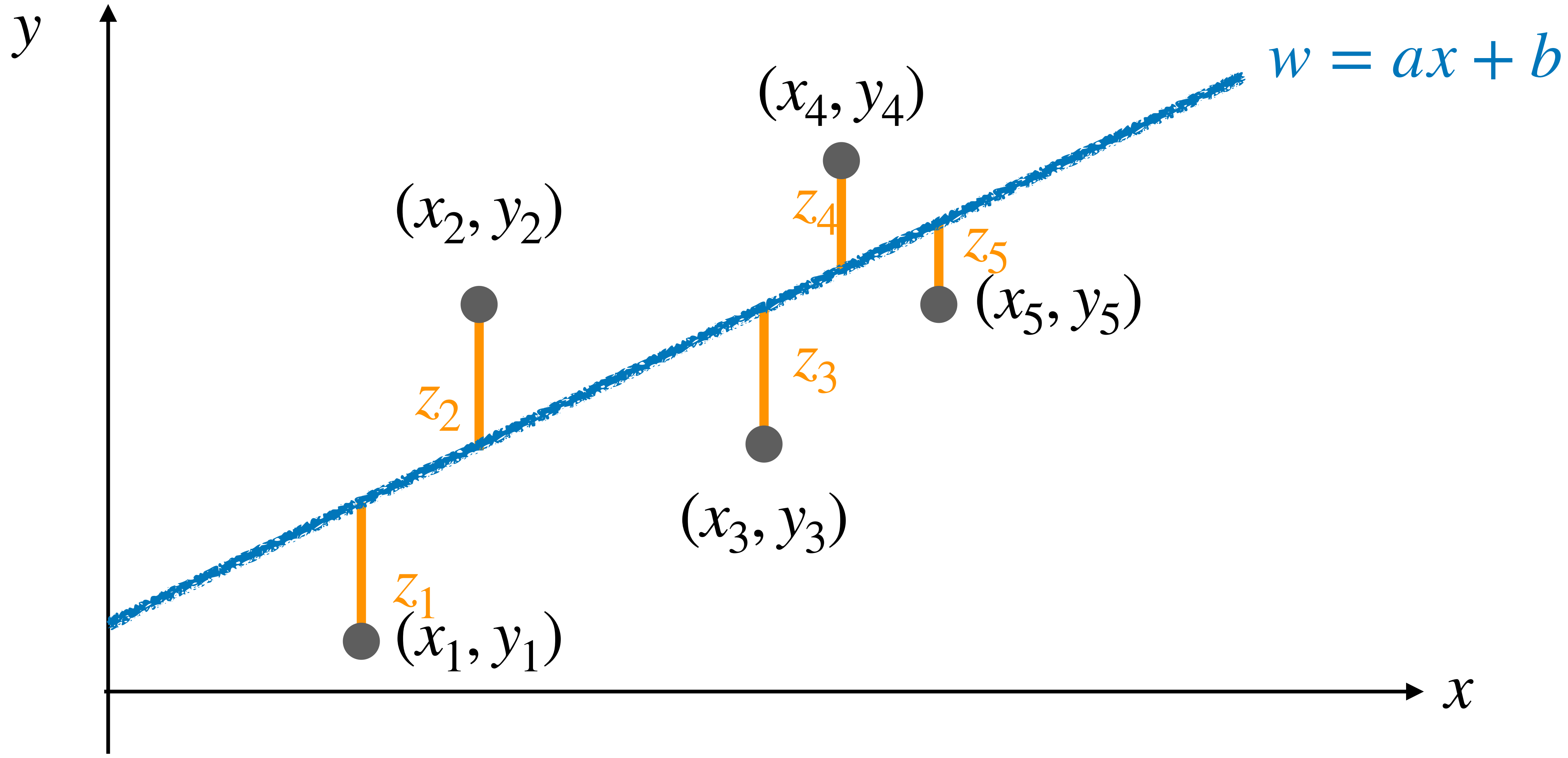

Find a line to minimize $\sum_{i=1}^{n} z_i$

such that $z_i = |ax_i + b - y_i|$

Absolute values

Minimize $\sum_{j \in J} c_j |x_j|$ $\quad (c_j > 0)$

s.t. $\sum_{j \in J} a_{ij} x_j \geq b_i$ for all i

x_j is free for all j

- Replace x_j by $x_j^+ - x_j^-$, where $x_j^+ \geq 0$ and $x_j^- \geq 0 \Rightarrow |x_j| = x_j^+ + x_j^-$

Absolute values

LP

Minimize $\sum_{j \in J} c_j \, |x_j| \qquad (c_j > 0)$

s.t. $\sum_{j \in J} a_{ij} \, x_j \geq b_i$ for all i

x_j is free for all j

Minimize $\sum_{j \in J} c_j \, (x_j^+ + x_j^-) \qquad (c_j > 0)$

s. t. $\sum_{j \in J} a_{ij} \, (x_j^+ - x_j^-) \geq b_i$ for all i

$x_j^+, x_j^- \geq 0$ for all j

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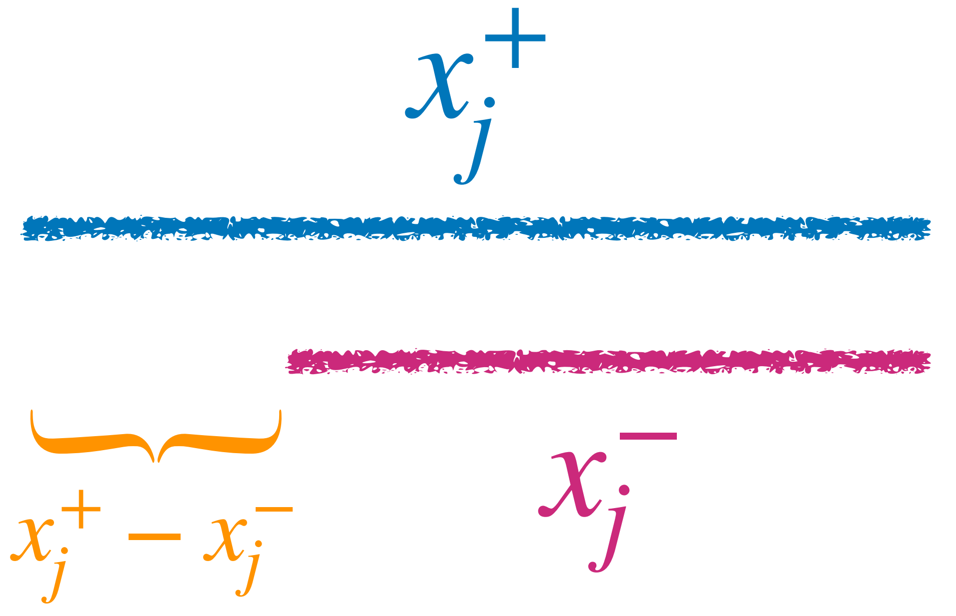

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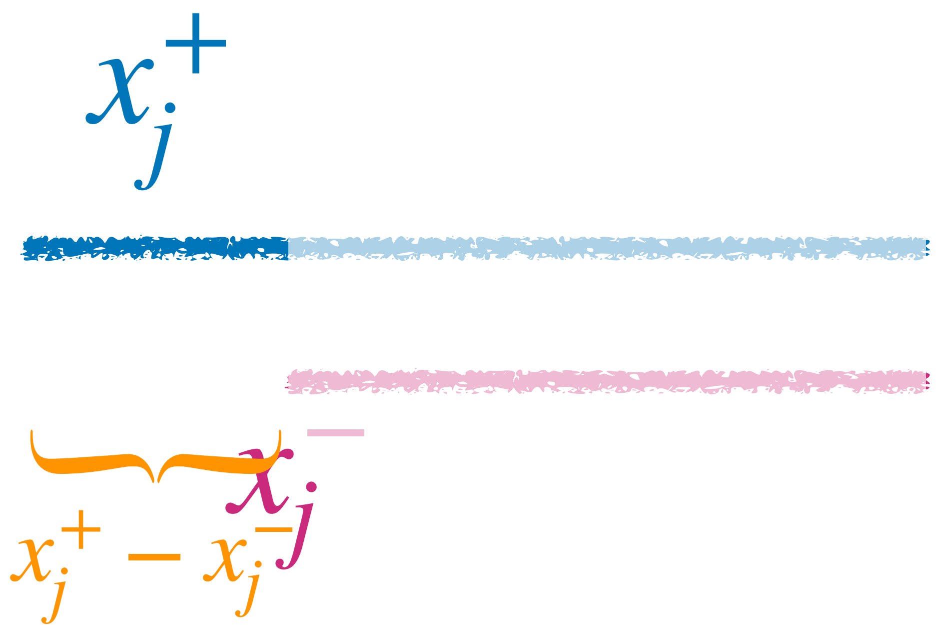

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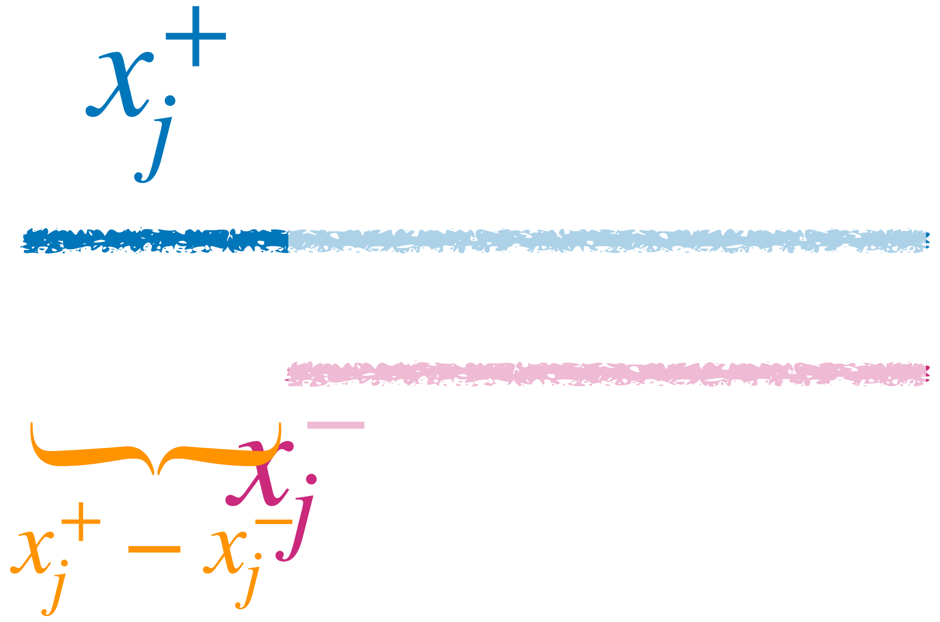

$$\text{Minimize} \sum_{j \in J} c_j \, |x_j| \qquad (c_j > 0)$$

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LP

$$\text{Minimize} \sum_{j \in J} c_j (x_j^+ + x_j^-) \downarrow \qquad (c_j > 0)$$

does not change

$$\text{s. t.} \sum_{j \in J} a_{ij} (x_j^+ - x_j^-) \geq b_i \text{ for all } i$$

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    $x_j = x_j^+$ when $x_j \geq 0$; $x_j = -x_j^-$ when $x_j \leq 0$

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Absolute values

- Replace the variable x who's absolute value is considered by $x^+ - x^-$
  - x^+ is the amount of positive part, and x^- the amount of negative part
  - The solution's optimality automatically forces at least one of x^+ and x^- be 0

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Min-max objective

- Consider the diet choice problem, where each type of nutrient i has a minimum amount needed m_i and price p_i, and there is an amount of budget B

Min-max objective

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- What happens if we concern about the maximum cost spent on a type of nutrient?

Min-max objective

- Consider the diet choice problem, where each type of nutrient i has a minimum amount needed m_i and price p_i, and there is an amount of budget B
- What happens if we concern about the maximum cost spent on a type of nutrient?
  - That is, we want to minimize $\max_i \{p_i \cdot x_i\}$ while satisfying the constraints

Min-max objective

P

Minimize $\max_{k \in K} \sum_{j \in J} c_{kj} x_j$

s. t. $\sum_{j \in J} a_{ij} x_j \geq b_i$ for all i

$x_j \geq 0$

Min-max objective

P

Minimize $\max_{k \in K} \sum_{j \in J} c_{kj}\, x_j$

s. t. $\sum_{j \in J} a_{ij}\, x_j \geq b_i$ for all i

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- Let $z = \max_{k \in K} \sum_{j \in J} c_{kj} x_j$

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- Let $z = \max_{k \in K} \sum_{j \in J} c_{kj} x_j$
  - That is, every possible $\sum_{j \in J} c_{kj} x_j$ should not be larger than z

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Minimize $\max_{k \in K} \sum_{j \in J} c_{kj}\, x_j$

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  - Then, we only need to minimize z

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s. t. $\sum_{j \in J} a_{ij} x_j \geq b_i$ for all i

$x_j \geq 0$

LP

Minimize z

s. t. $\sum_{j \in J} a_{ij} x_j \geq b_i$ for all i

$\sum_{j \in J} c_{kj} x_j \leq z$ for all k

$x_j \geq 0$ for all j

- Let $z = \max_{k \in K} \sum_{j \in J} c_{kj} x_j$
  - That is, every possible $\sum_{j \in J} c_{kj} x_j$ should not be larger than z
  - Then, we only need to minimize z

Min-max objective

- Introduce a new variable z that represents the maximum value of the targeted variable x
  - Relate z with all the possible value of the targeted variable x by restricting $x \leq z$ in any case

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Discontinuous-values variables

- Consider that you are a manager of a store and need to manage the amount of items in the store so the items are always available
- However, the provider of item x has a range-constraint on every purchase: whenever you buy item x, the amount must be in $[\ell, u]$
    - That is, $x = 0$ or $\ell \leq x \leq u$

Discontinuous-values variables

Minimize $\sum_{j \in J} c_j x_j$

s. t. *(constraints)*

$x_j = 0$ or $\ell \leq x_j \leq u \; \forall j \in J'$

Minimize $\sum_{j \in J} c_j x_j$

s. t. *(constraints)*

$x_j \leq u \cdot y_j$ for all $j \in J'$

$x_j \geq \ell \cdot y_j$ for all $j \in J'$

$y_j \in \{0,1\}$ for all $j \in J'$

- Introduce a binary *indicator variable* $y_j \in \{0,1\}$ (hope: $y_j = 0$ if $x_j = 0$ and $y_j = 1$ if $x_j > 0$)
  - Observation: $x_j \leq u \cdot y_j$ and $x_j \geq \ell \cdot y_j$ whether $y_j = 0$ or $y_j = 1$

Discontinuous-values variables

- Introduce a binary *indicator variable* y
  - Hopefully, the value of y indicates different scenarios of choice of x
    - Need to relate the value of y and the value of x

Outline

- Warm up: Minimum spanning tree
- Tricks:
  - Range constraints
  - Absolute value objective
  - Min-max objective
  - Discontinuous-values variables
  - Fixed-cost objective
    - **Facility location**
    - Lot-sizing
  - Or and conditional conditions
- Solving ILP: Cutting plane

Facility Location

- When the objective value is discontinuous

Facility Location

- Given a set of potential depots $N = \{1, \cdots, n\}$ and a set $M = \{1, \cdots, m\}$ of clients, suppose that the use of depot j associates with a fixed cost f_j, and there is a transportation cost c_{ij} if one unit of the demand of client i is served by depot j. The problem is to decide which depots to open, and which depot serves each client so as to minimize the sum of the fixed and transportation cost

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depots j

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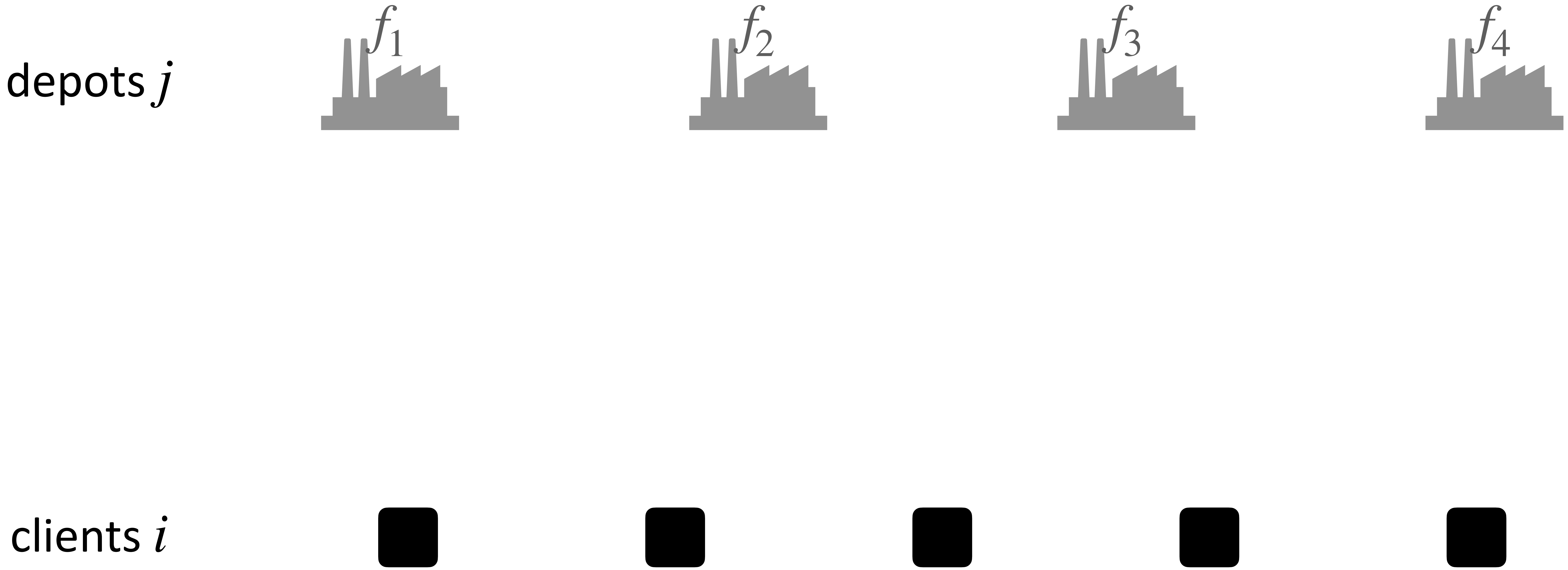

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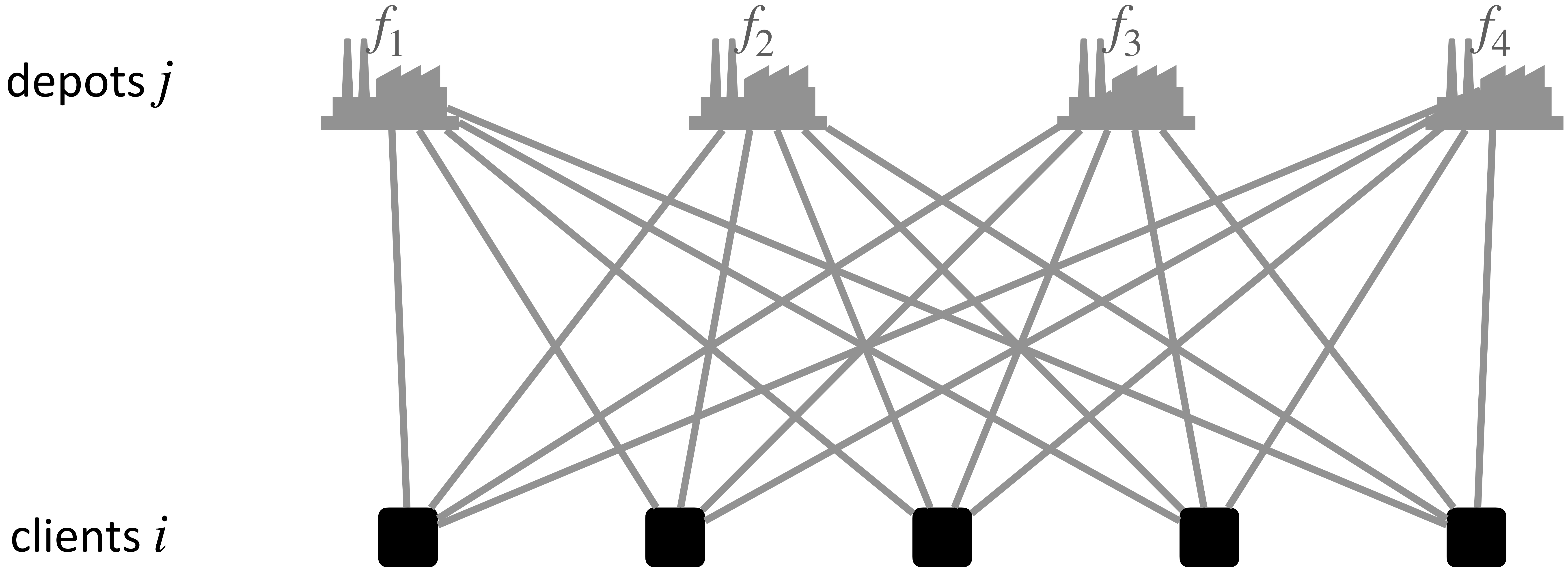

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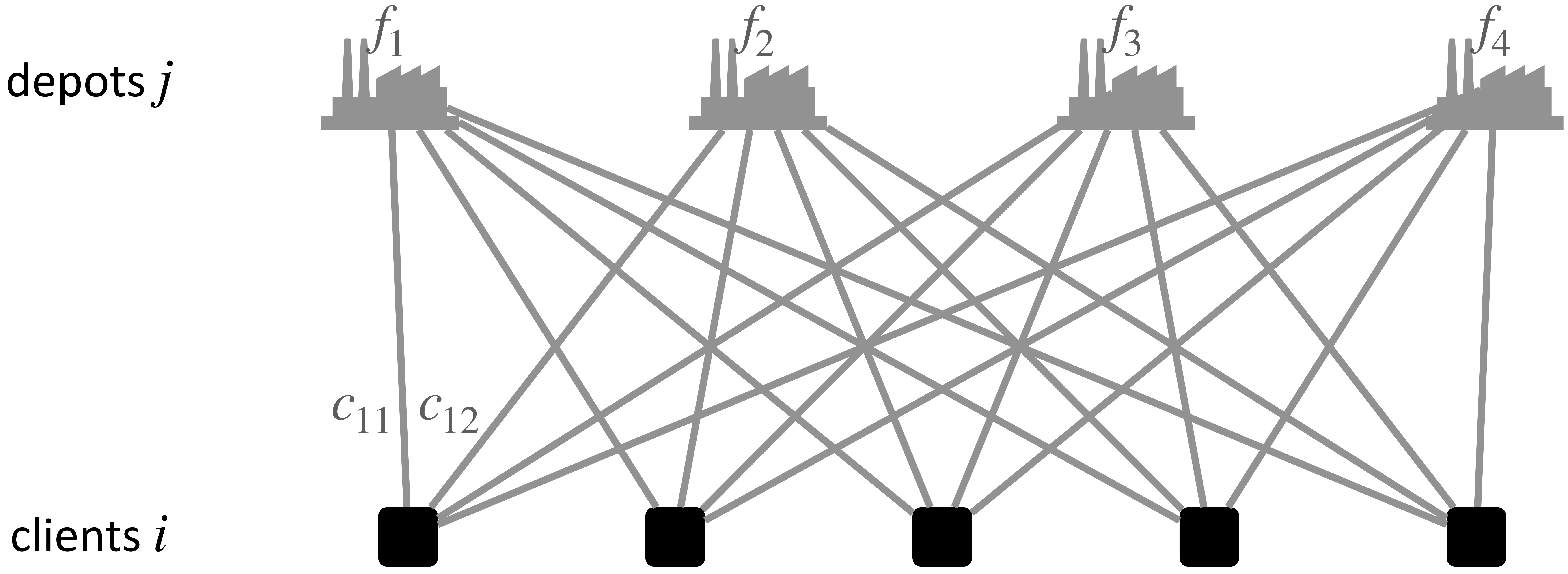

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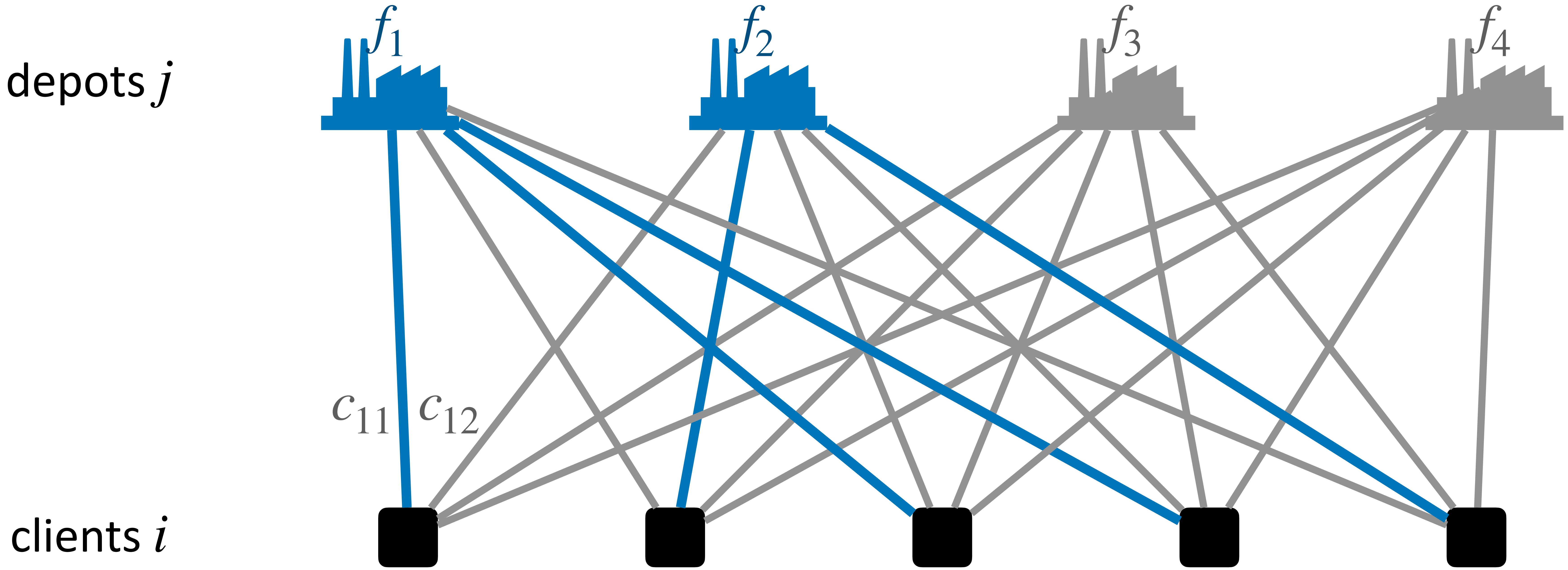

cost $= f_1 + f_2 + c_{11} + c_{22} + c_{31} + c_{41} + c_{52}$

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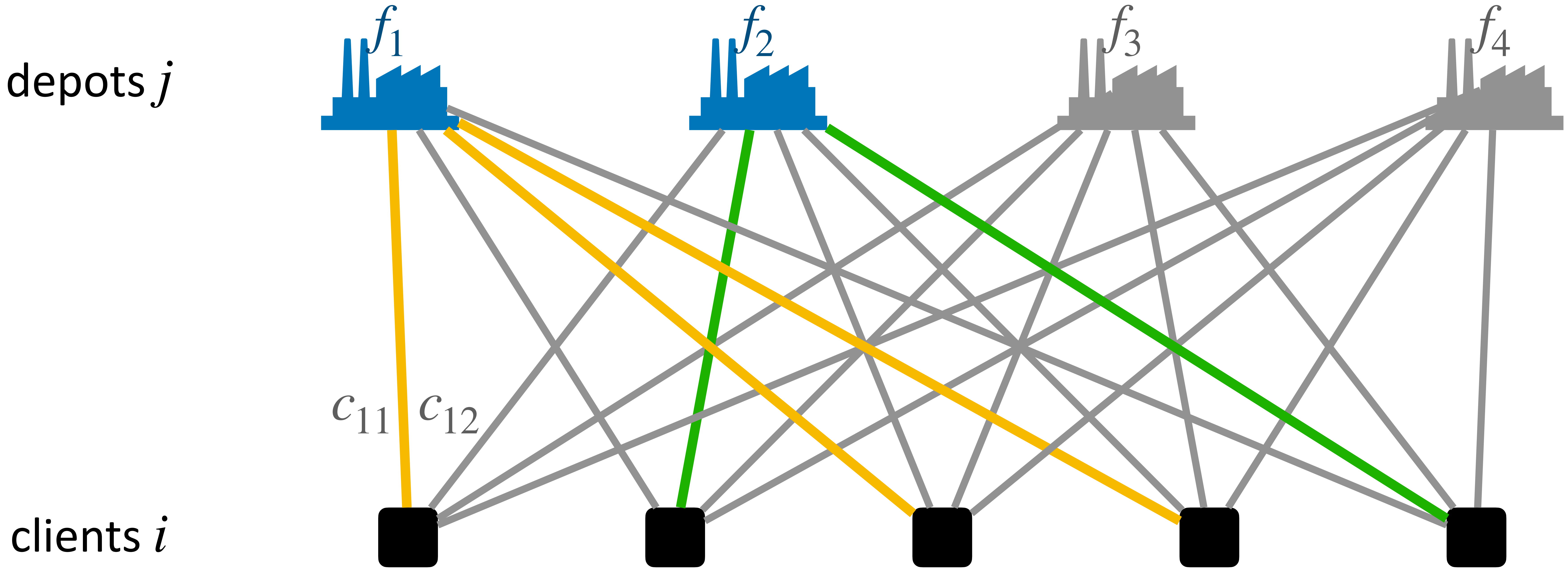

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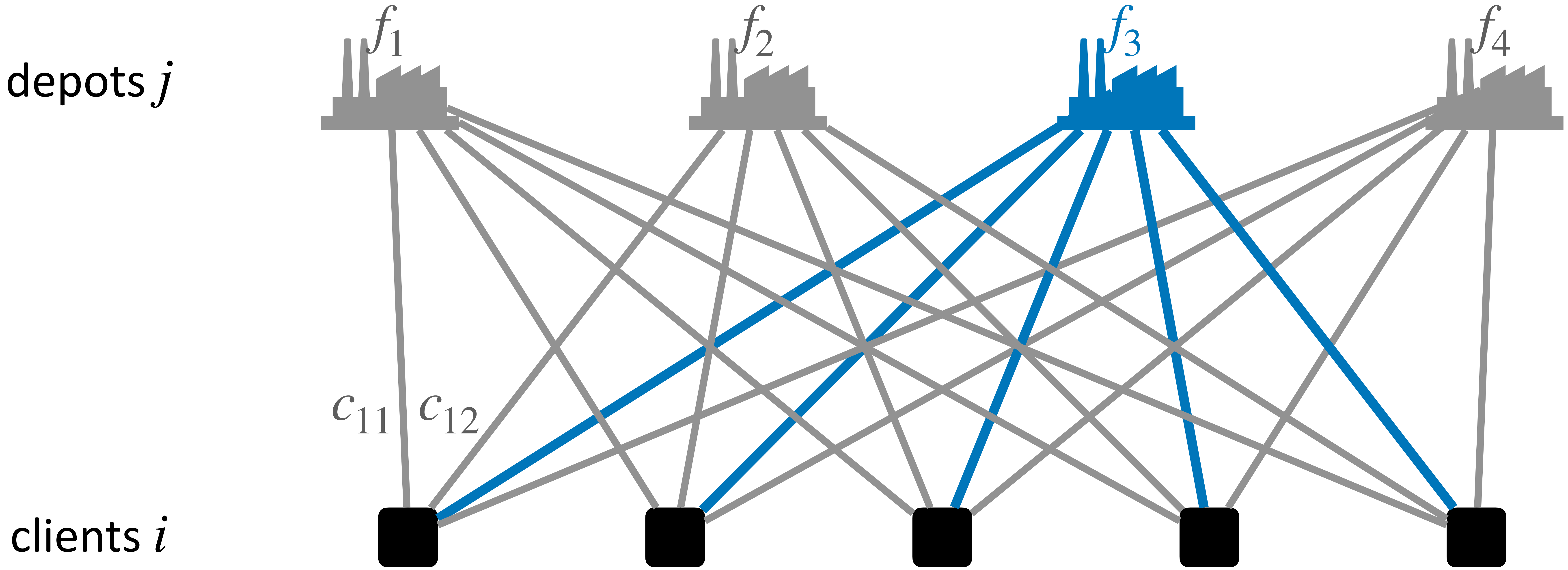

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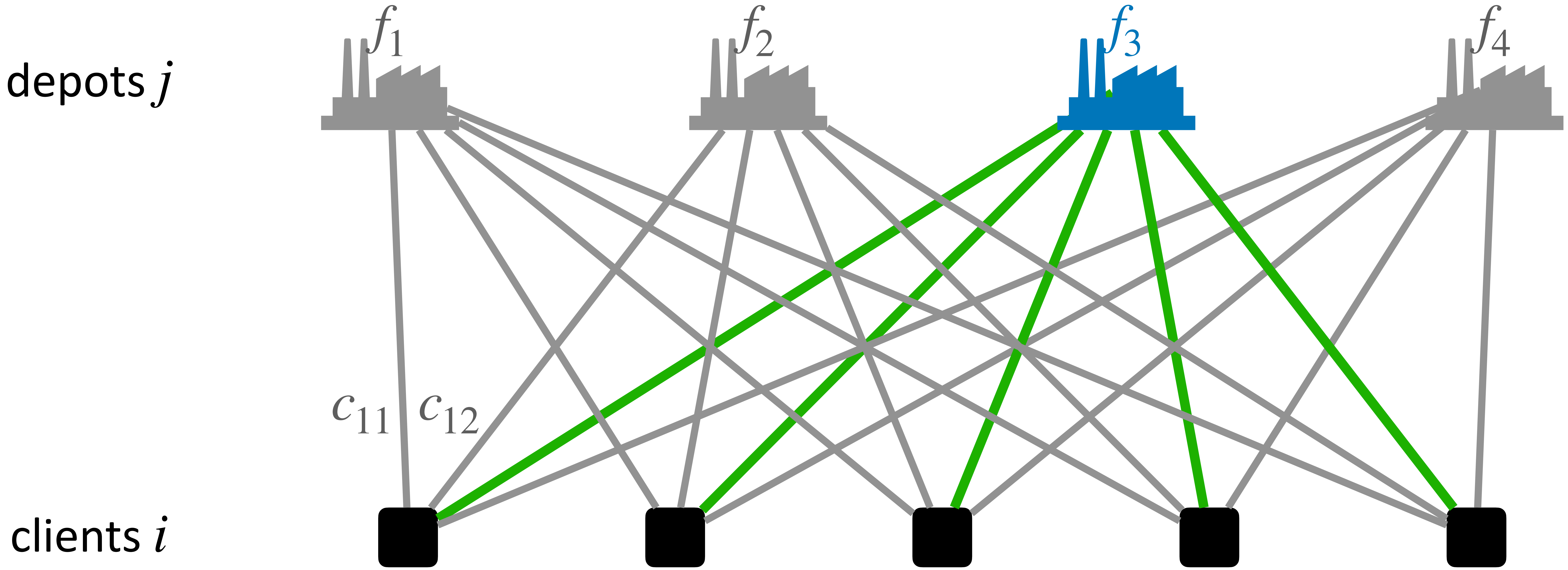

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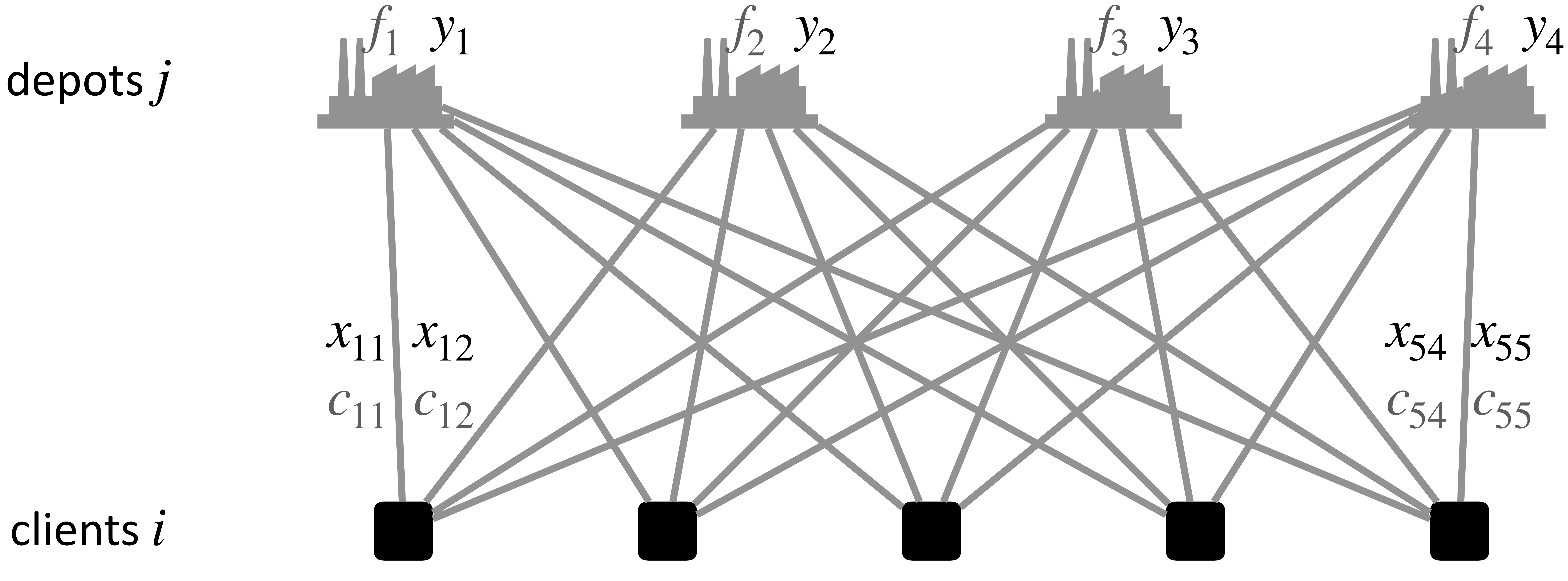

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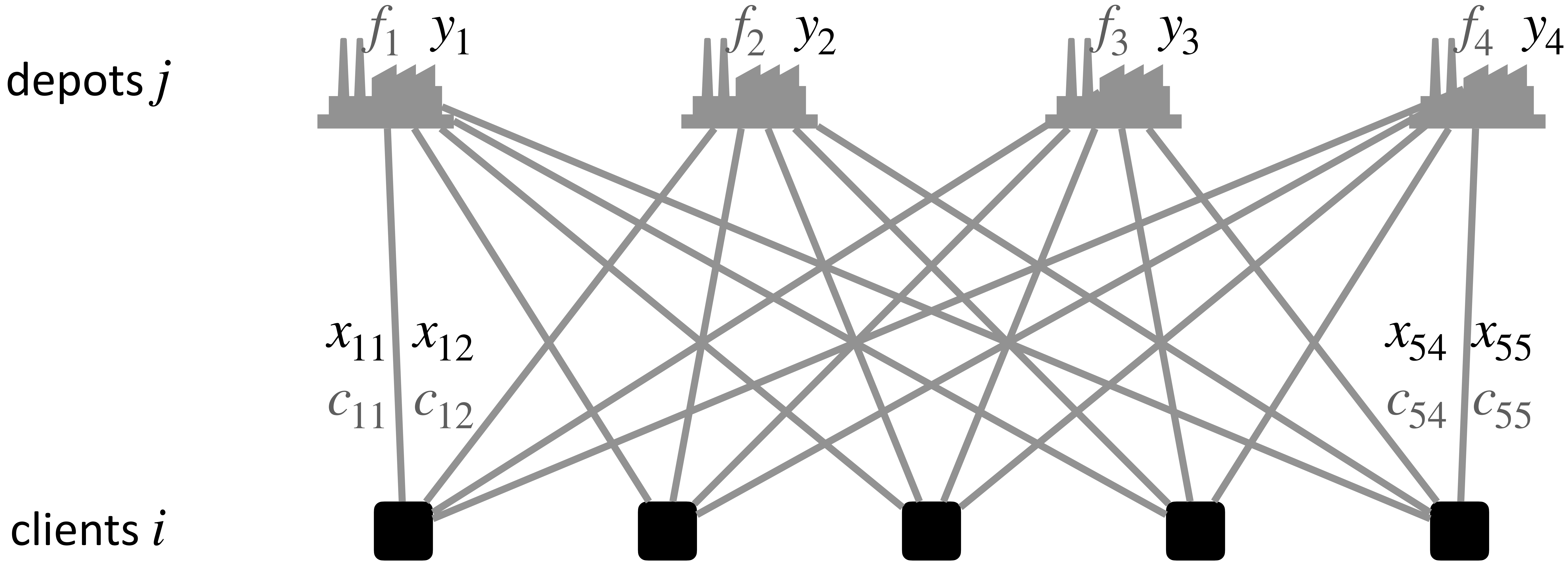

min $\Sigma_{j=1}^{n} y_j + \Sigma_{j=1}^{n} \Sigma_{i=1}^{m} c_{ij} x_{ij}$

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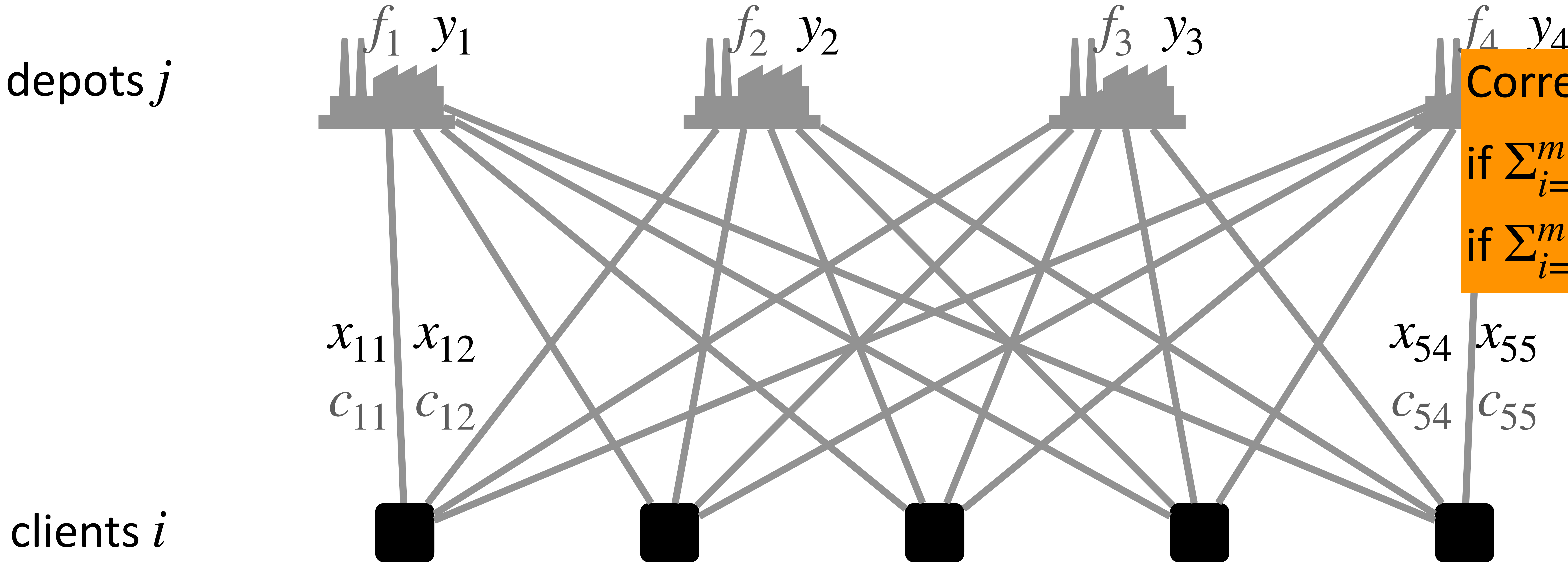

Correlation of y_j and $\Sigma_{i=1}^{m} x_{ij}$:

if $\Sigma_{i=1}^{m} x_{ij} = 0, y_j = 0$

if $\Sigma_{i=1}^{m} x_{ij} > 0, y_j = 1$

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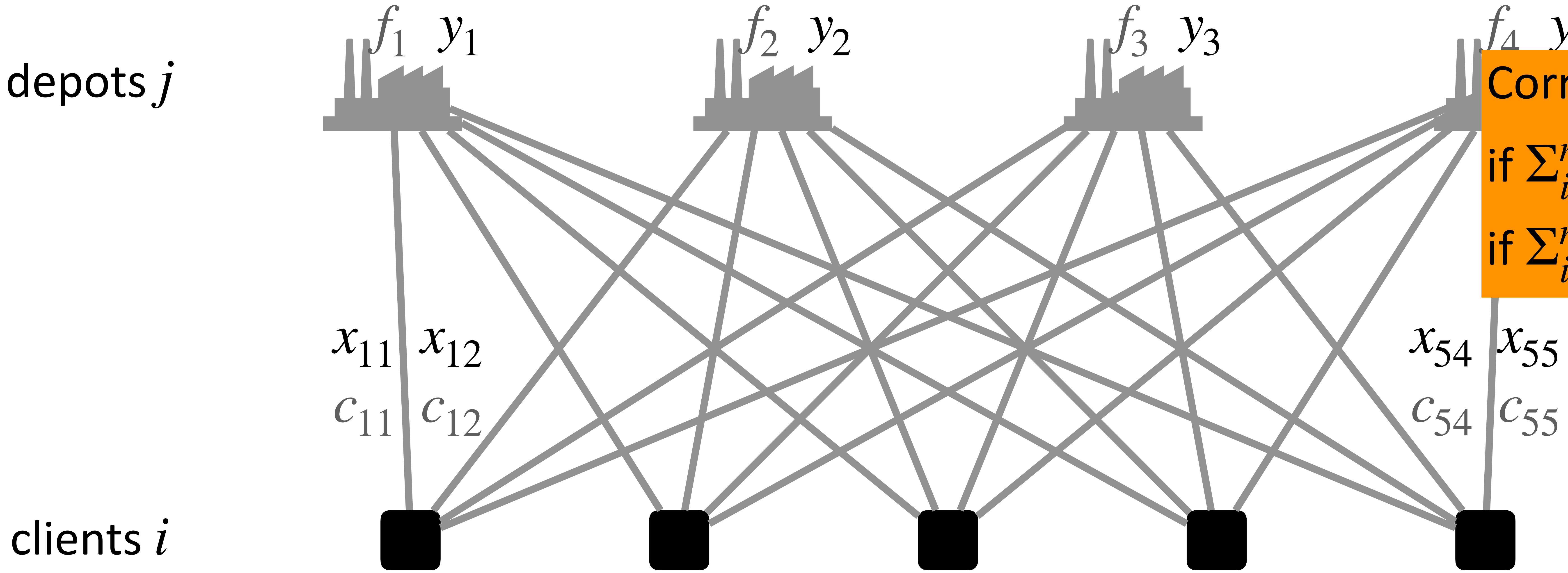

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$\Sigma_{i=1}^m x_{ij} \leq m \cdot y_j$ for all j

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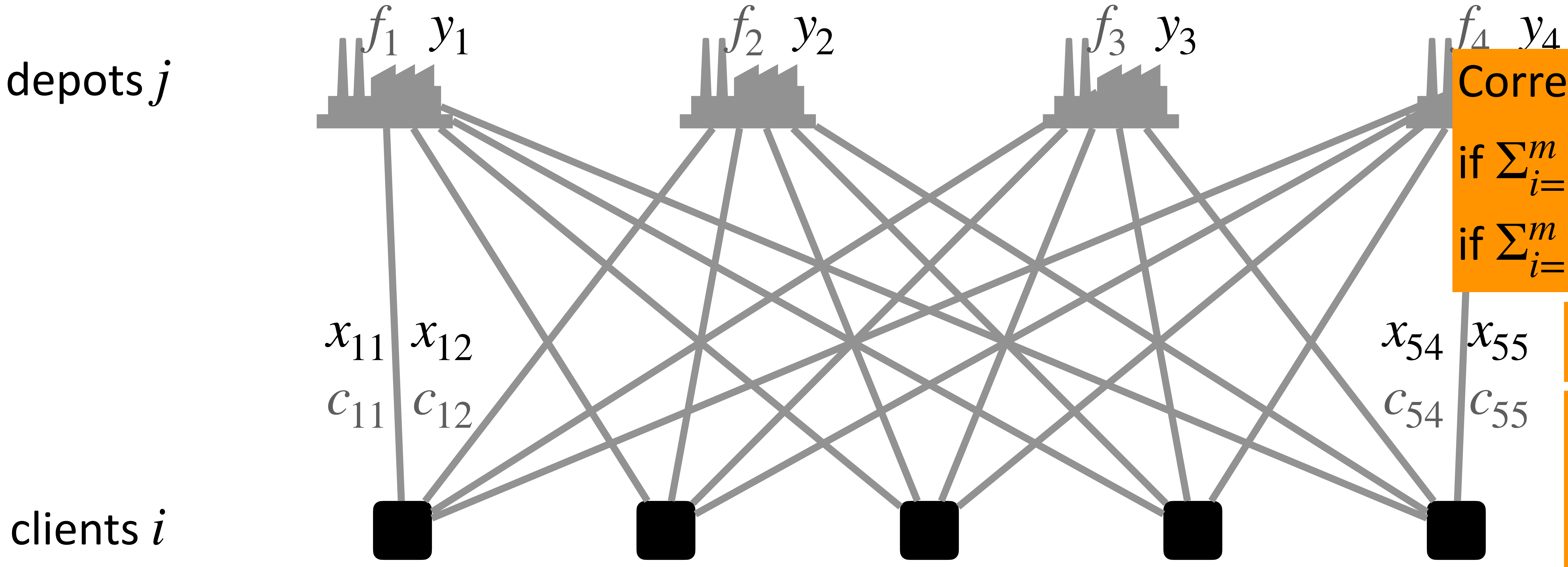

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$\Sigma_{i=1}^m x_{ij} \le m \cdot y_j$ for all j

Alternatively:
if $x_{ij} = 1, y_j = 1$
if $x_{ij} = 0, y_j = 0$ or 1

min $\Sigma_{j=1}^n y_j + \Sigma_{j=1}^n \Sigma_{i=1}^m c_{ij} x_{ij}$

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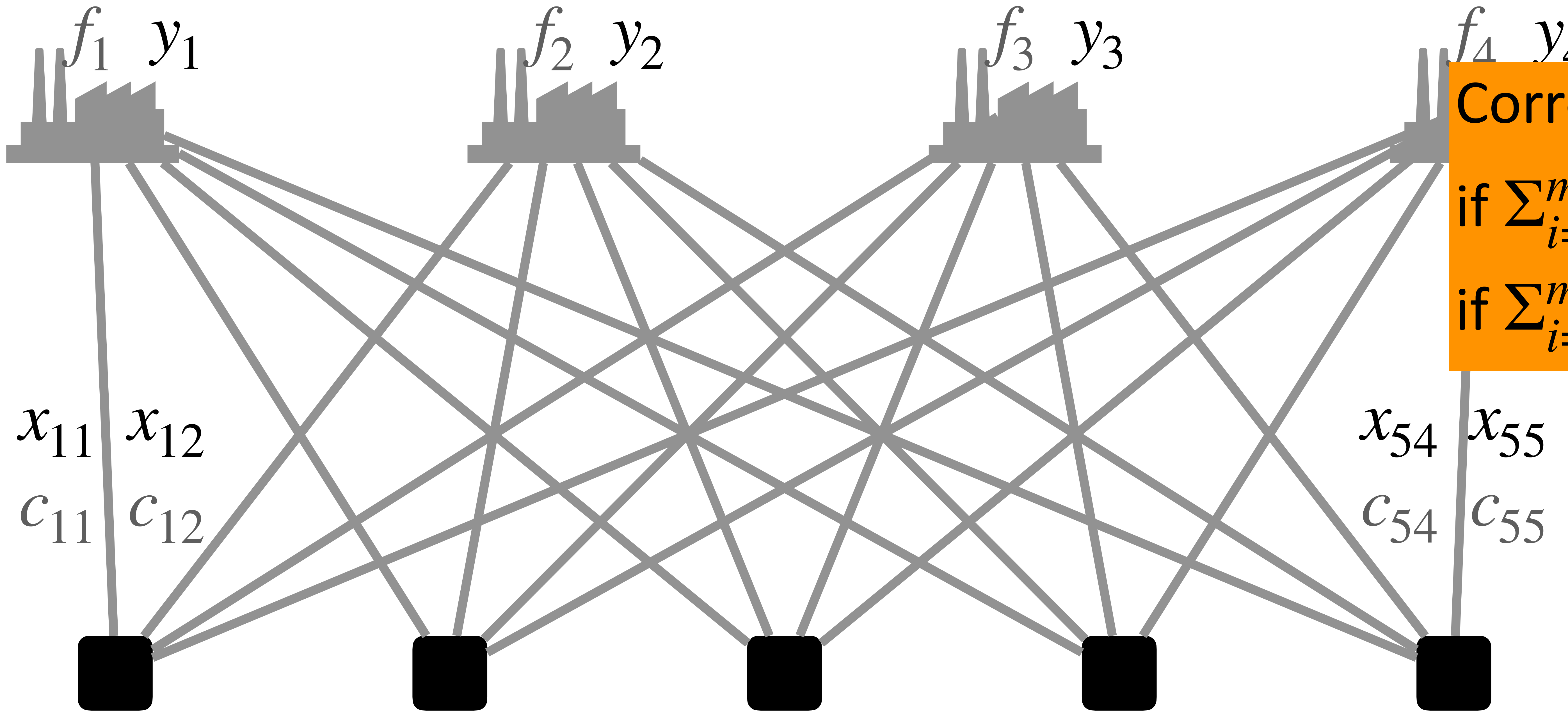
depots j



clients i

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if $x_{ij} = 0, y_j = 0$ or 1

$x_{ij} \leq y_j$ for all i and j

min $\Sigma_{j=1}^{n} y_j + \Sigma_{j=1}^{n} \Sigma_{i=1}^{m} c_{ij} x_{ij}$

Facility Location

- Variables:
  - For every depot j, the variable $y_j = 1$ if j is used, and $y_j = 0$ otherwise
  - $x_{ij} = 1$ if the demand of client i satisfied from depot j, and $x_{ij} = 0$ otherwise
- minimize $\Sigma_{j=1}^{n} f_j y_j + \Sigma_{i=1}^{m} \Sigma_{j=1}^{n} c_{ij} x_{ij}$

  subject to $\Sigma_{j=1}^{n} x_{ij} = 1$ for $i = 1, \cdots, m$

  $\Sigma_{i=1}^{m} x_{ij} \leq m y_j$ for $j = 1, \cdots, n$

  $x_{ij} \geq 0$ for $i = 1, \cdots, m, j = 1, \cdots, n$

  $y_j \in \{0,1\}$ for $j = 1, \cdots, n$

Outline

- Warm up: Minimum spanning tree
- Tricks:
  - Range constraints
  - Absolute value objective
  - Min-max objective
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  - **Fixed-cost objective**
    - Facility location
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Fixed cost

Minimize $F(x)$

s.t. $\sum_{j \in J} a_{ij} w_j \geq b_i$ for all i

$x \geq 0$

$w_j \geq 0$ for all j

- where $F(x) = 0$ for $x = 0$, and $F(x) = k + cx$ for $x > 0$

Minimize $ky + cx$

s.t. $\sum_{j \in J} a_{ij} w_j \geq b_i$ for all i

$x \leq uy$

$x \geq 0$

$w_j \geq 0$ for all j

$y \in \{0,1\}$

- Introduce a binary indicator variable $y \in \{0,1\}$ ($y = 0$ for $x = 0$, and $y = 1$ for $x > 0$)
  - Relate y and the objective function in different choices of x

Tips

- Use a binary indicator variable $y \in \{0,1\}$ ($y = 0$ for $x = 0$, and $y = 1$ for $x > 0$) to indicate the objective value under different choices of x

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Lot-Sizing

- On day t, there is a demand of d_t, a fixed producing cost of f_t, a production cost of p_t per unit of production, and storage cost of h_t per unit of production. The problem is to decide on a production plan for an n-day horizon for a single product.

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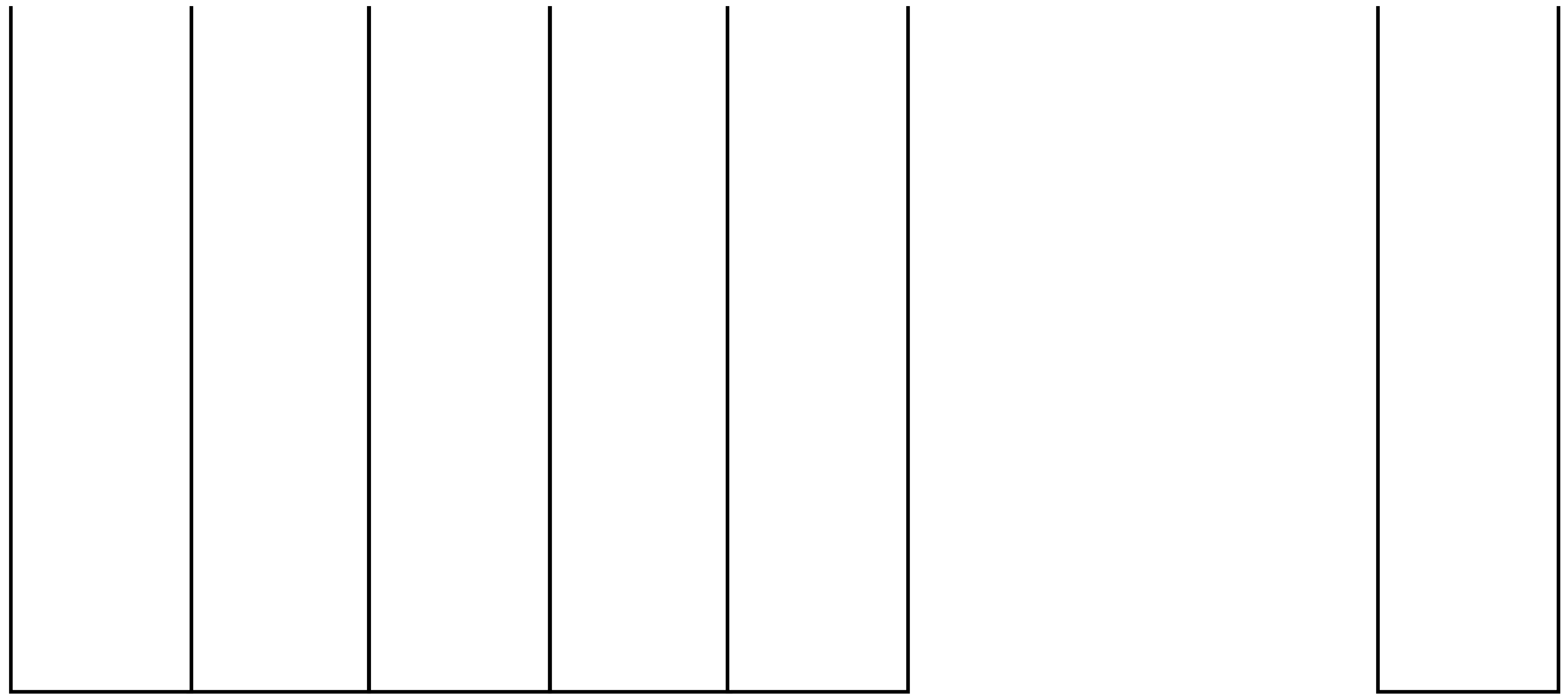

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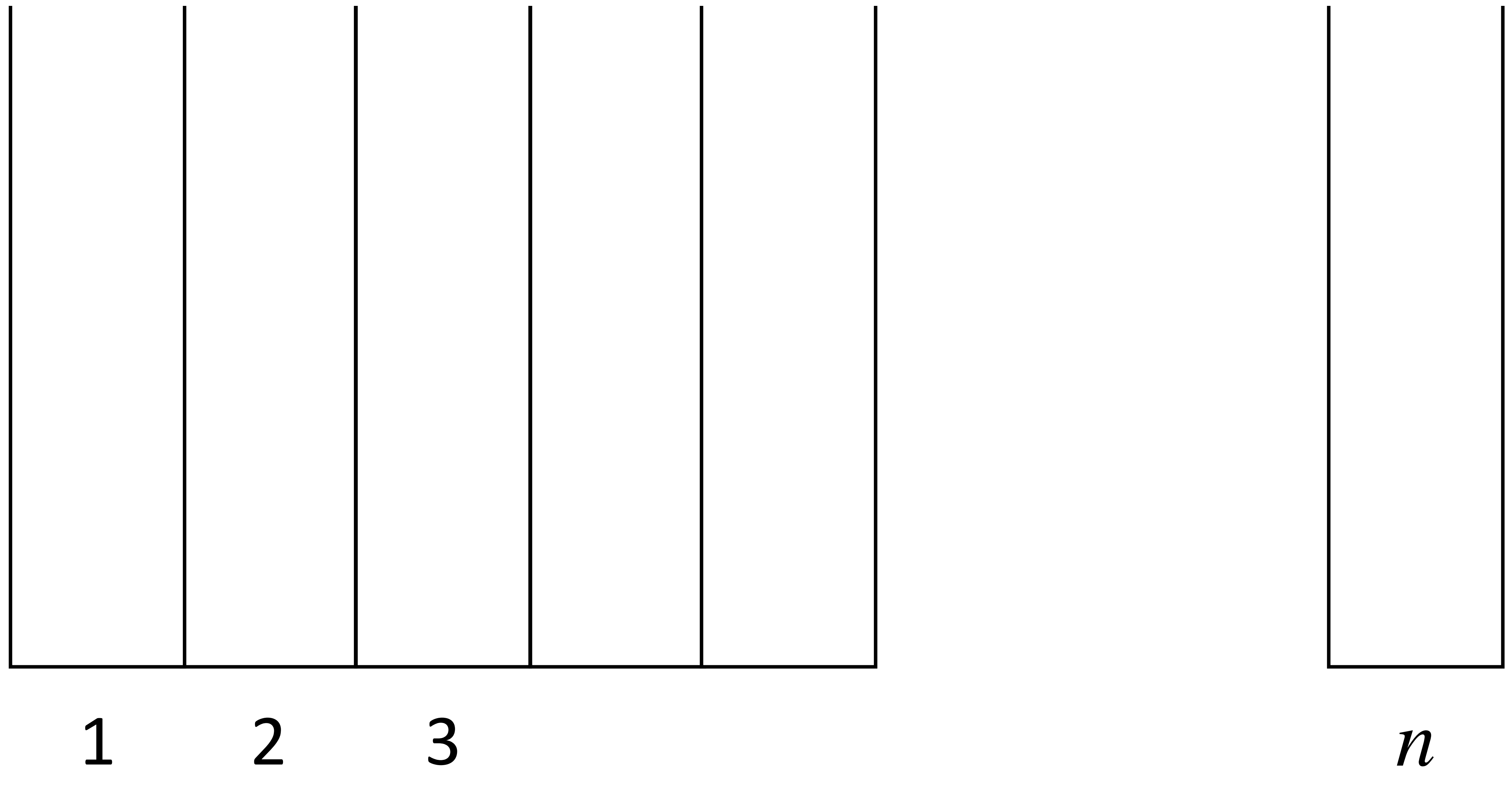

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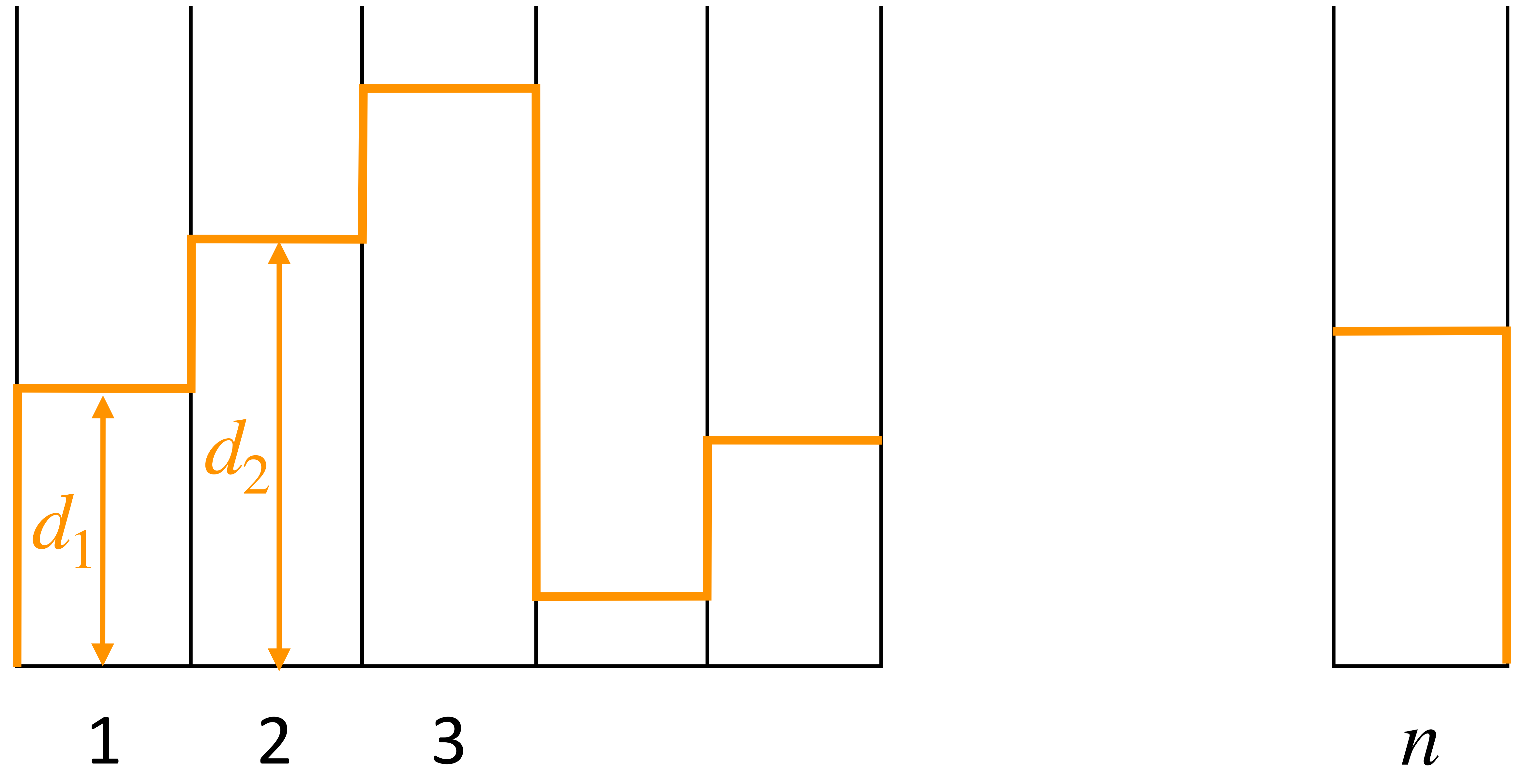

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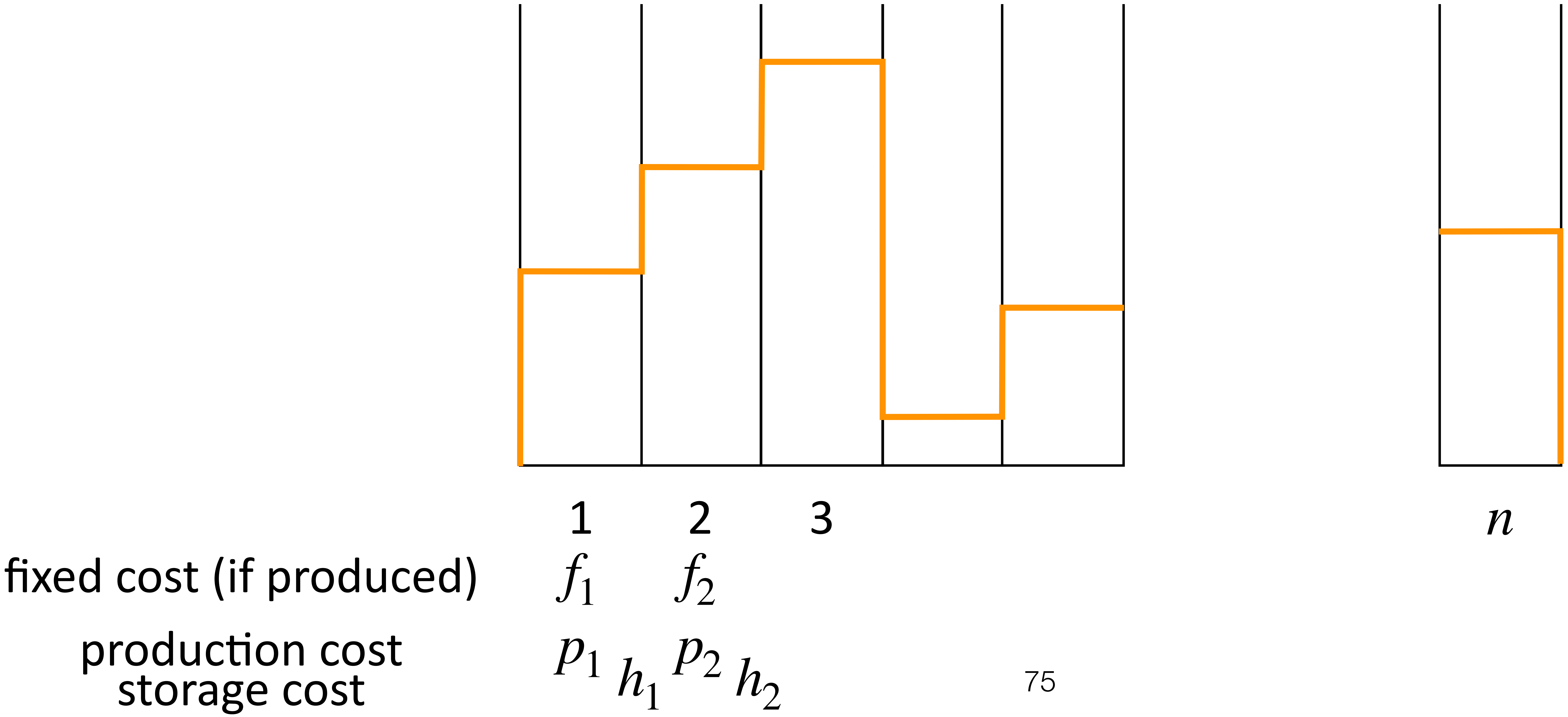

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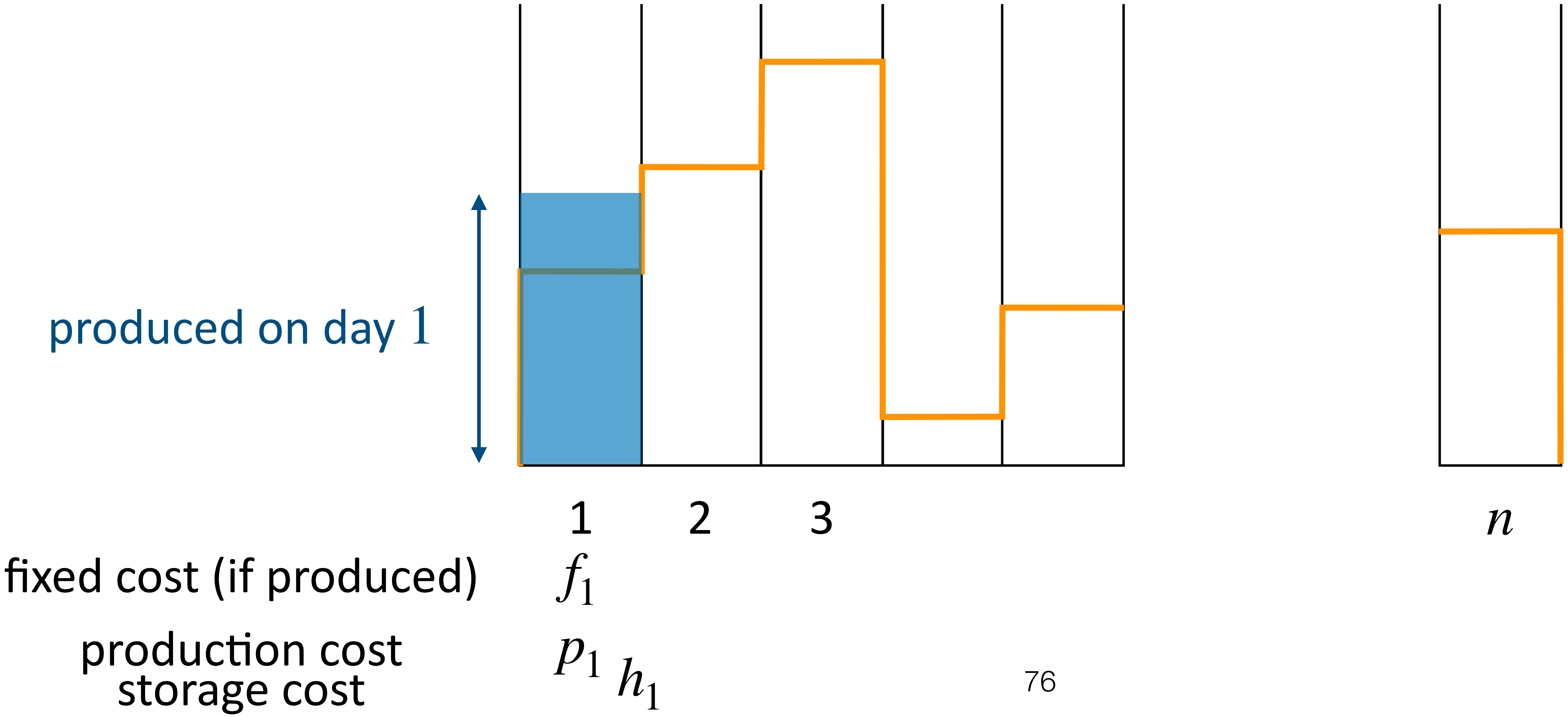

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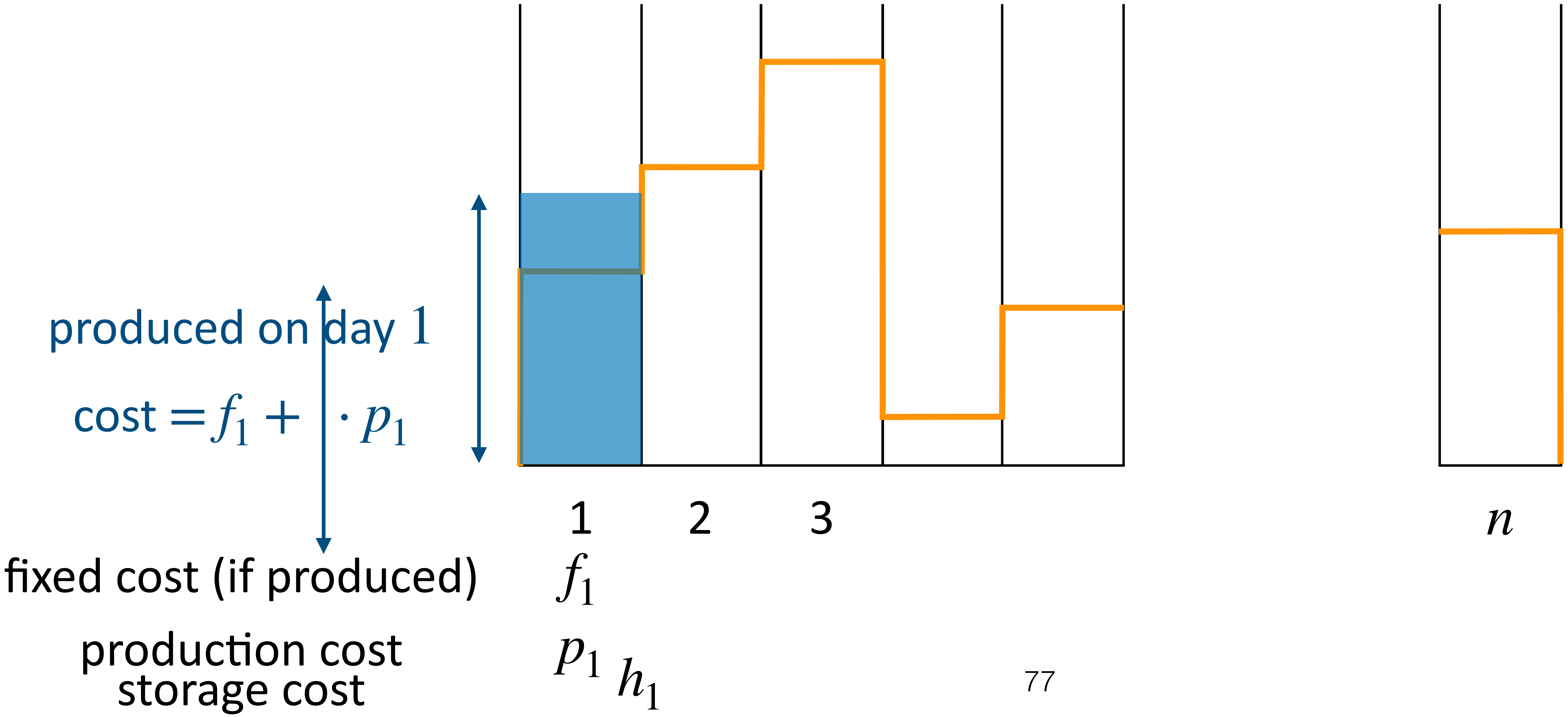

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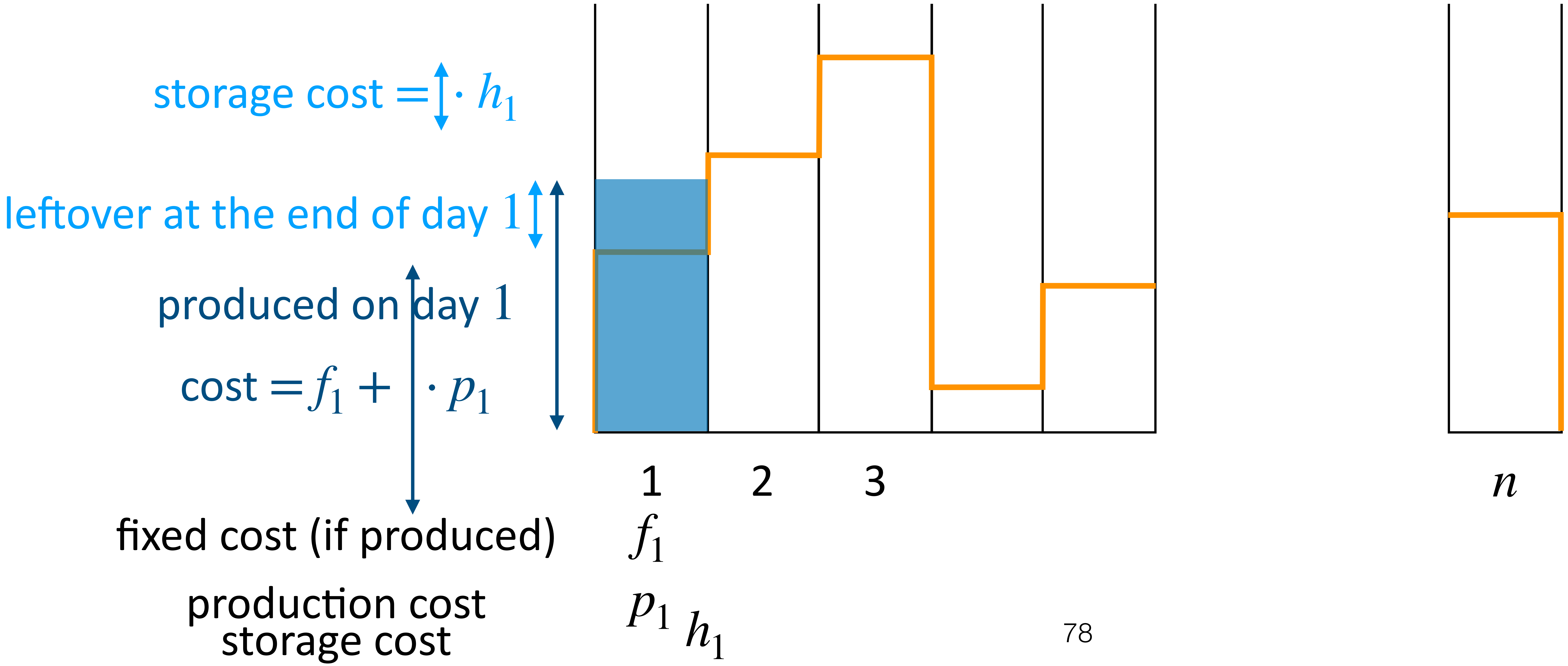

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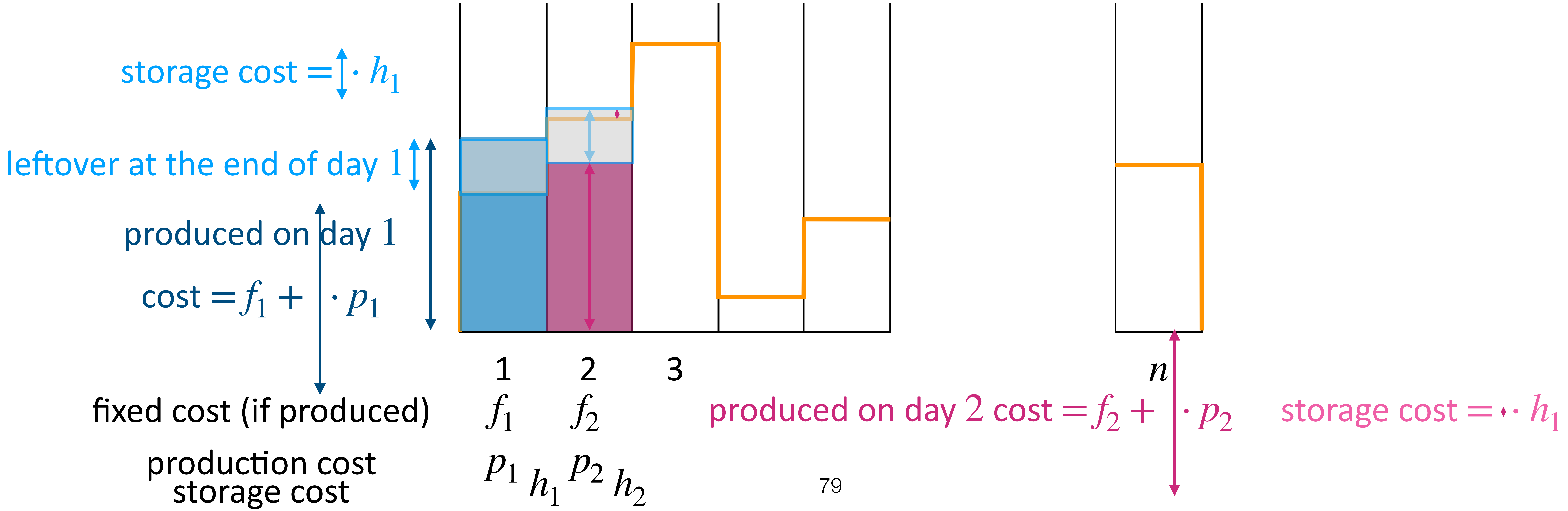

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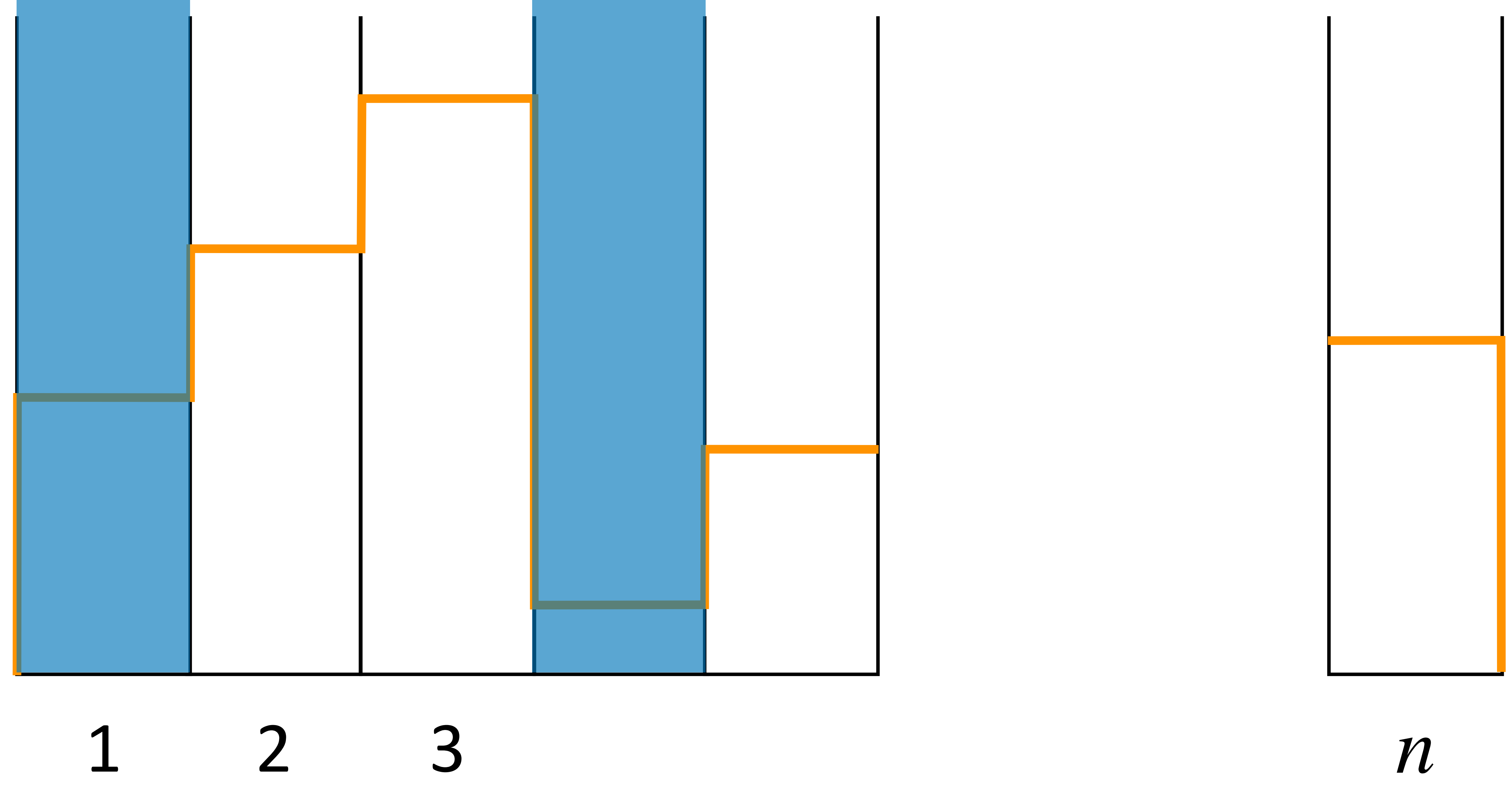

fixed cost (if produced) f_1 f_2

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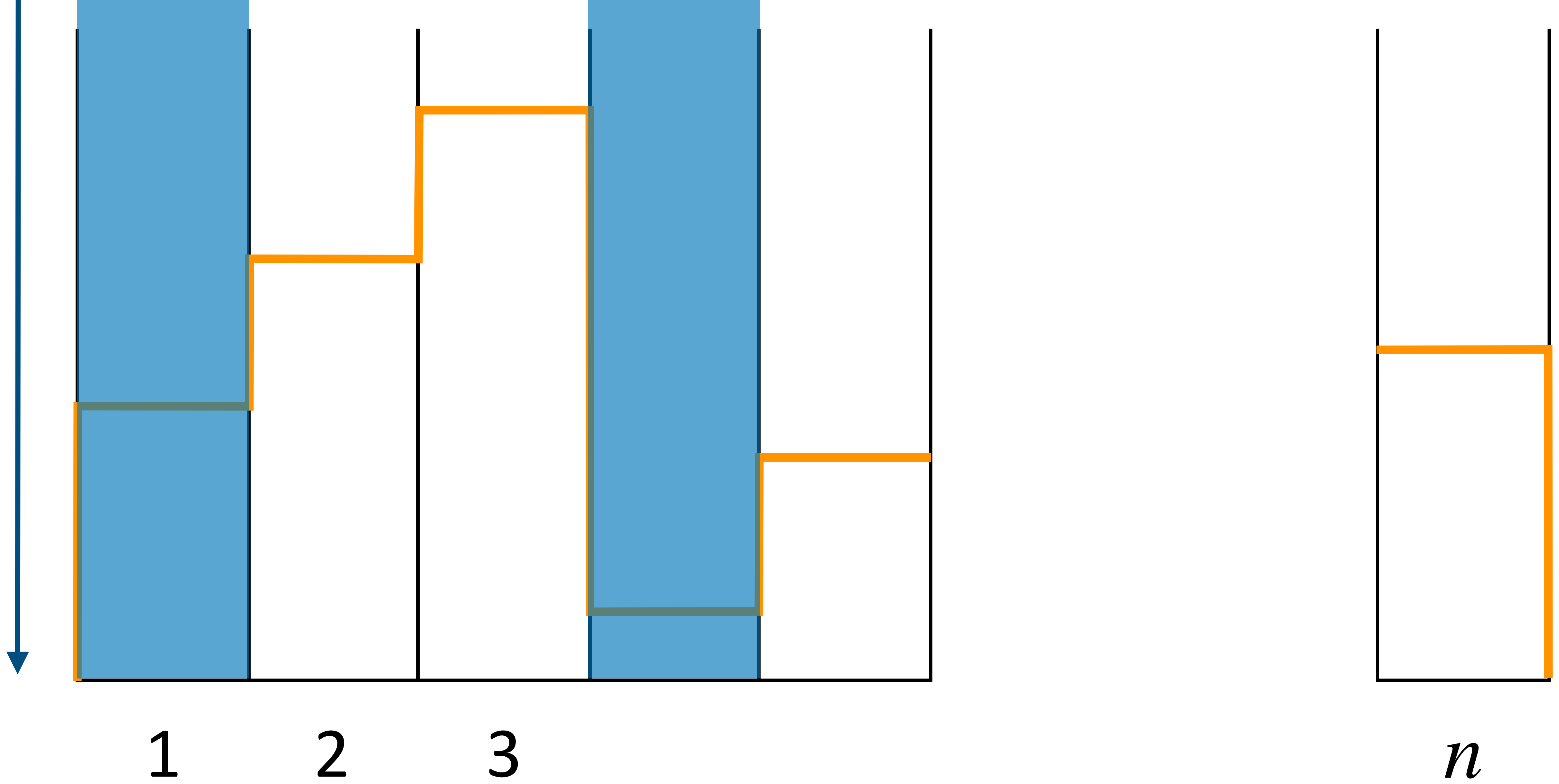

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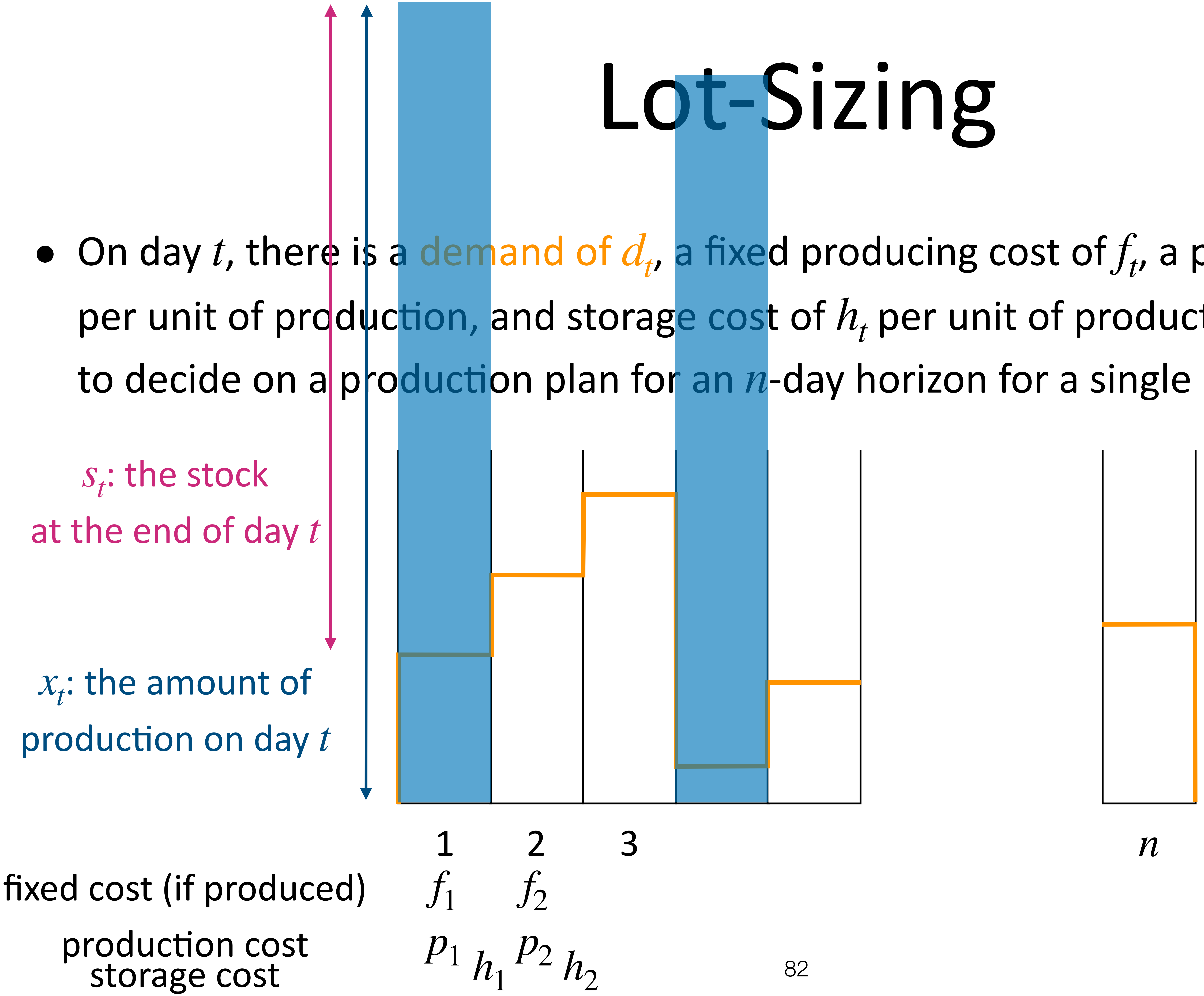

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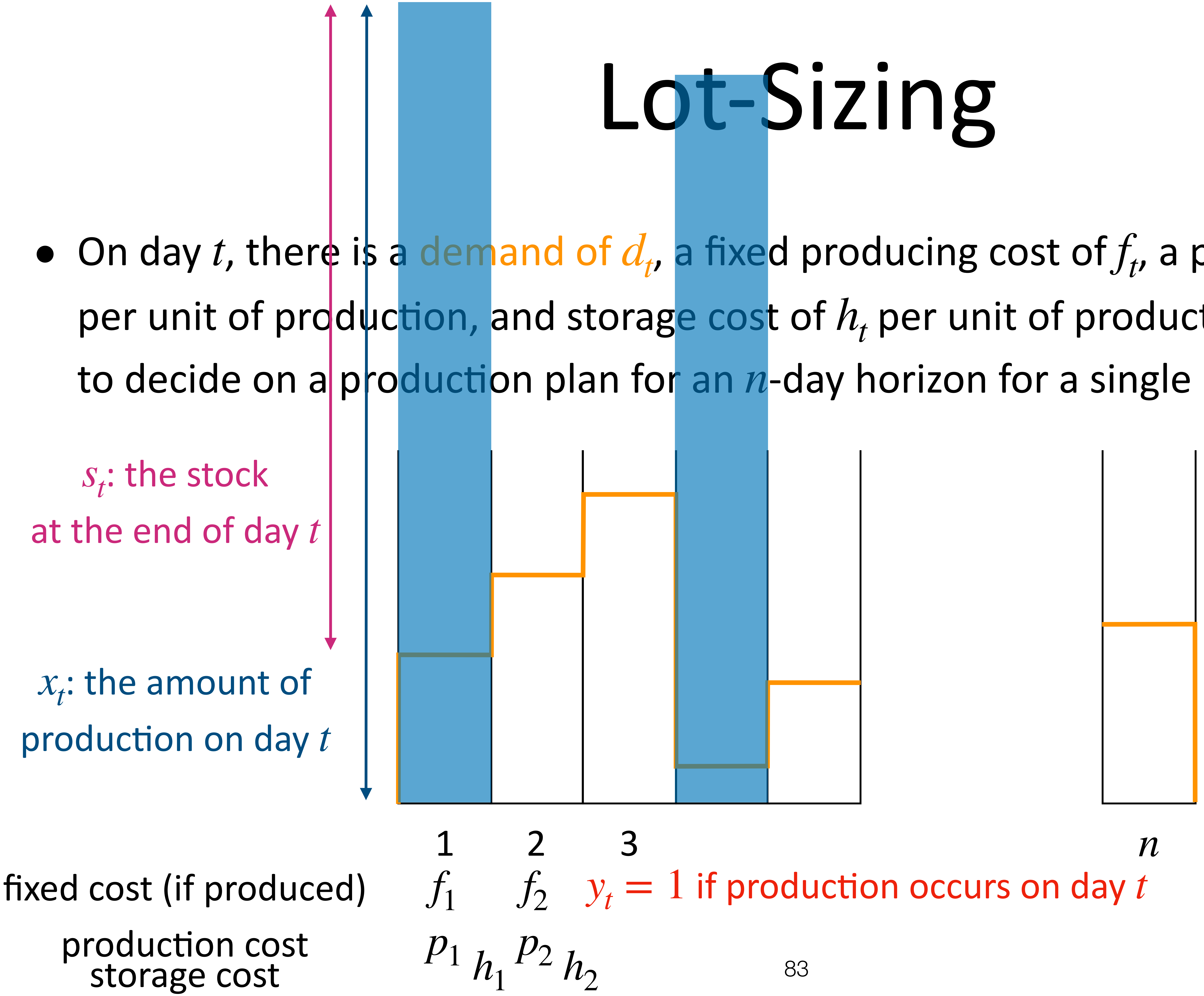

$y_t = 1$ if production occurs on day t

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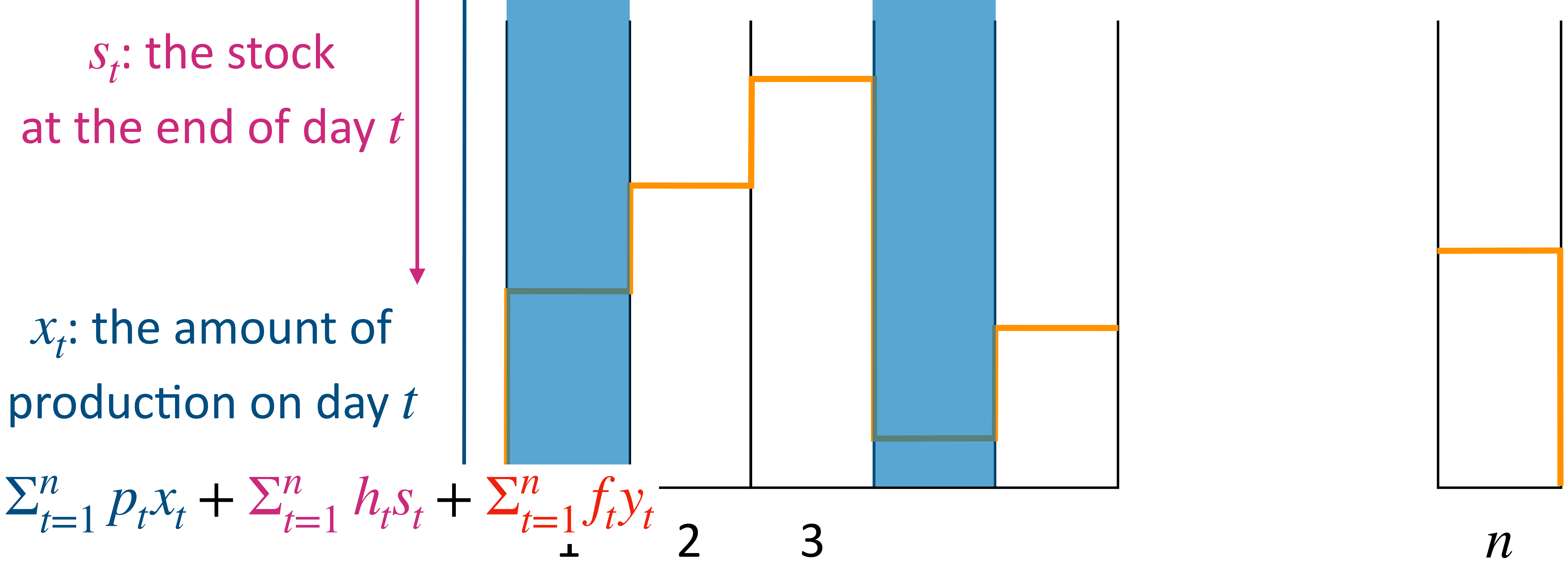

min $\Sigma_{t=1}^{n} p_t x_t + \Sigma_{t=1}^{n} h_t s_t + \Sigma_{t=1}^{n} f_t y_t$

fixed cost (if produced)	f_1		f_2	
production cost	p_1		p_2	
storage cost		h_1		h_2

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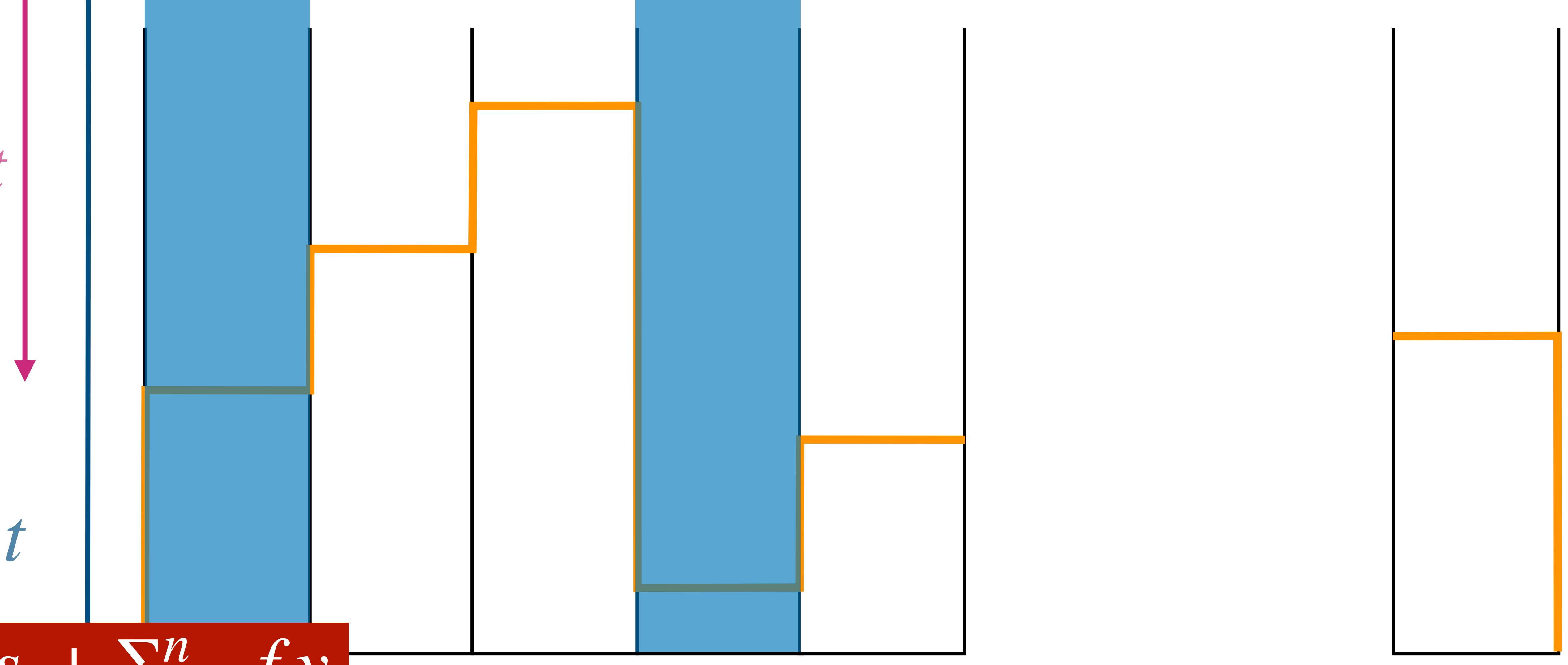

s_t: the stock at the end of day t

x_t: the amount of production on day t

min $\Sigma_{t=1}^{n} p_t x_t + \Sigma_{t=1}^{n} h_t s_t + \Sigma_{t=1}^{n} f_t y_t$

	1	2	3		n
fixed cost (if produced)	f_1	f_2			
production cost	p_1	p_2			
storage cost	h_1	h_2			

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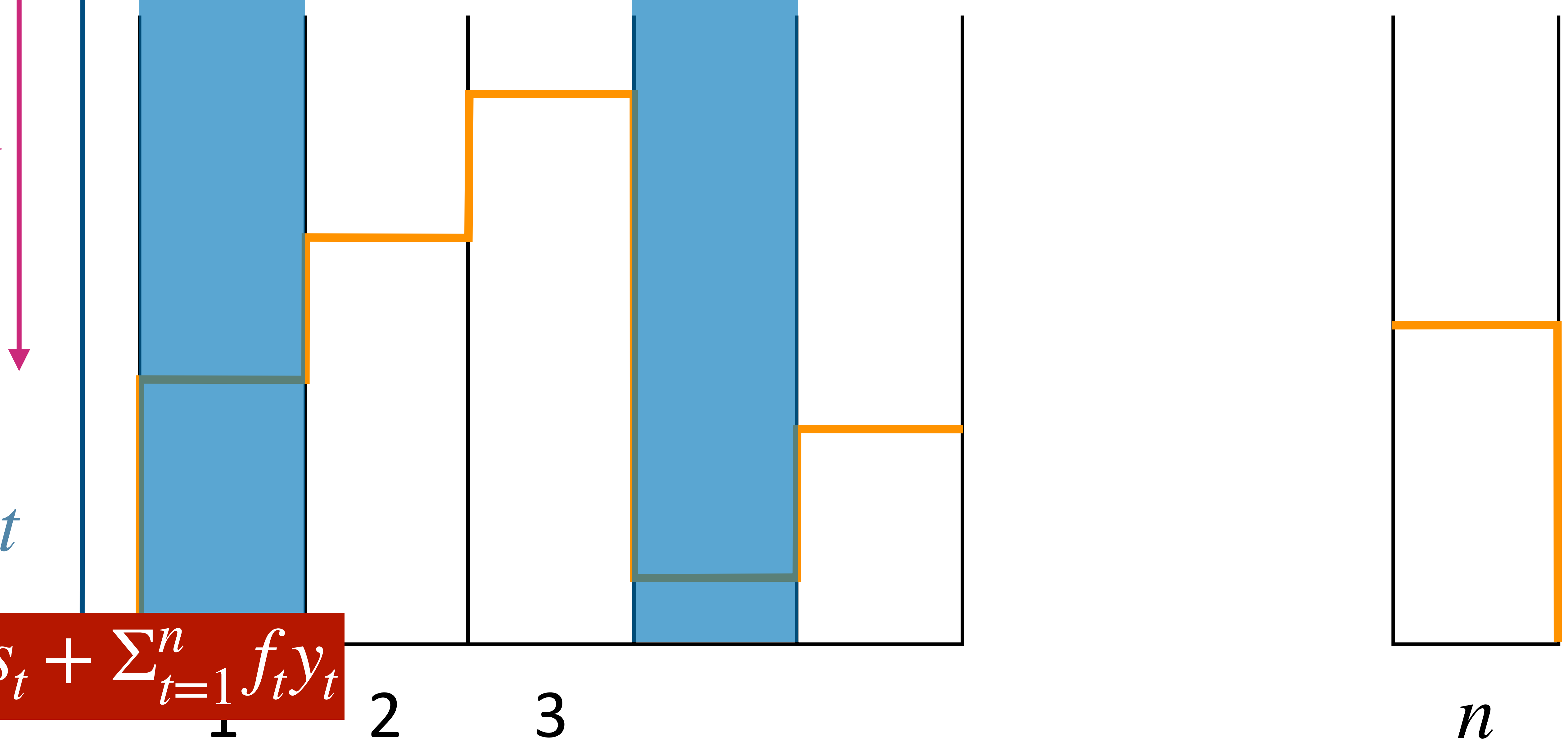

$\min \sum_{t=1}^{n} p_t x_t + \sum_{t=1}^{n} h_t s_t + \sum_{t=1}^{n} f_t y_t$

Every day, there should be enough (from production and saving) so the demand is satisfied

$x_t + (s_{t-1} - s_t) = d_t$ for all t

fixed cost (if produced) f_1 f_2

$y_t = 1$ if production occurs on day t

production cost p_1 p_2

storage cost h_1 h_2

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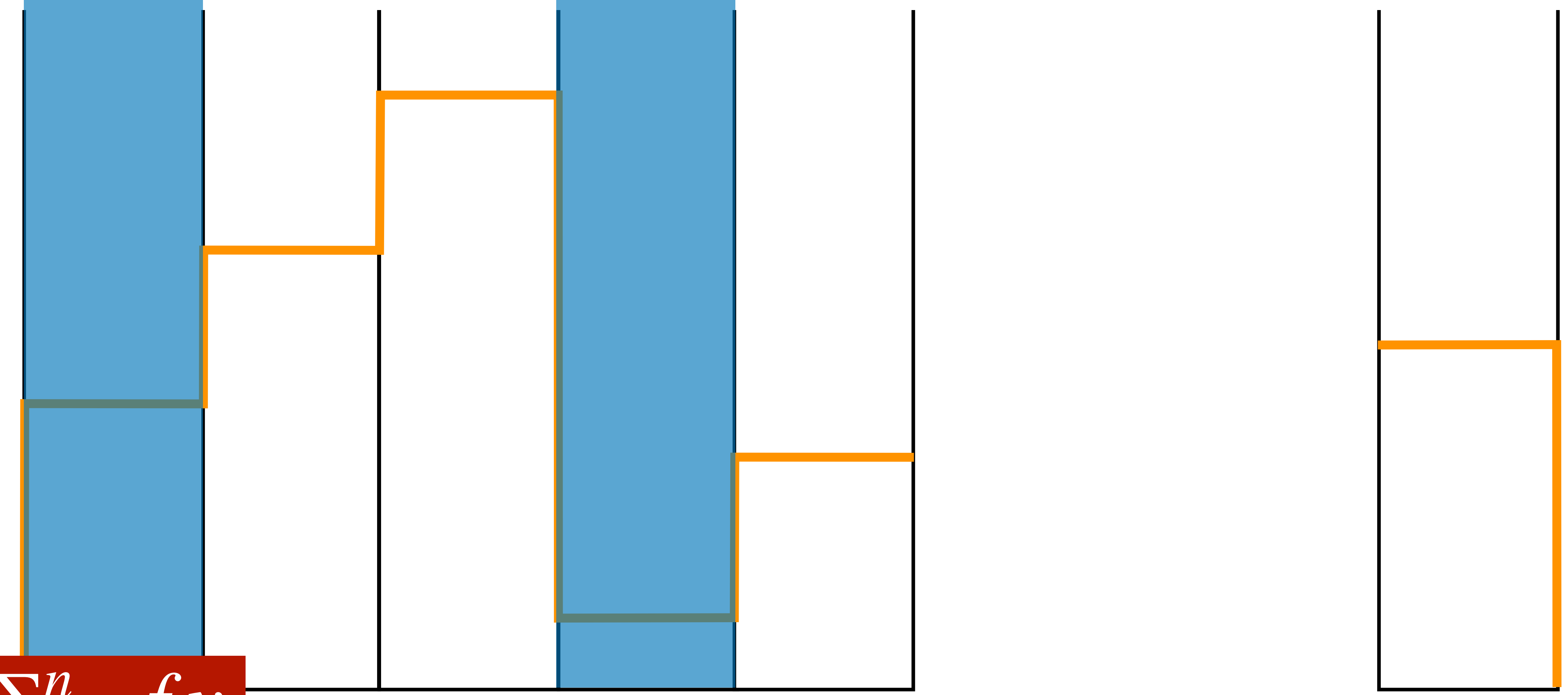

min $\Sigma_{t=1}^{n} p_t x_t + \Sigma_{t=1}^{n} h_t s_t + \Sigma_{t=1}^{n} f_t y_t$

	1	2	3	n
fixed cost (if produced)	f_1	f_2		
production cost	p_1	p_2		
storage cost	h_1	h_2		

$y_t = 1$ if production occurs on day t

Every day, there should be enough (from production and saving) so the demand is satisfied

$x_t + (s_{t-1} - s_t) = d_t$ for all t

Correlation of y_t and x_t:

if $x_t = 0, y_t = 0$

if $x_t > 0, y_t = 1$

$x_t \le y_t \cdot \Sigma_{t=1}^{n} d_t$ for all t

Lot-Sizing

- Variables:
  - x_t: the amount produced on day t
  - s_t: the stock at the end of day t
  - $y_t = 1$ if production occurs on day t, and $y_t = 0$ otherwise
- minimize $\Sigma_{t=1}^{n} p_t x_t + \Sigma_{t=1}^{n} h_t s_t + \Sigma_{t=1}^{n} f_t y_t$

  subject to $x_t + (s_{t-1} - s_t) = d_t$ for $t = 1,\cdots,n$

  $x_t \le y_t \cdot \Sigma_{t=1}^{n} d_t$ for $t = 1,\cdots,n$

  $s_0 = 0$

  $s_t, x_t \ge 0$ for $t = 1,\cdots,n$

  $y_t \in \{0,1\}$

Outline

- Warm up: Minimum spanning tree
- Tricks:
  - Range constraints
  - Absolute value objective
  - Min-max objective
  - Discontinuous-values variables
  - Fixed-cost objective
    - Facility location
    - Lot-sizing
  - Or and conditional conditions
- **Solving ILP: Cutting plane**

Different formulations of Facility Location

Different formulations of Facility Location

- Variables:
  - For every depot j, the variable $y_j = 1$ if j is used, and $y_j = 0$ otherwise
  - $x_{ij} = 1$ if the demand of client i satisfied from depot j, and $x_{ij} = 0$ otherwise
- minimize $\Sigma_{j=1}^{n} f_j y_j + \Sigma_{i=1}^{m} \Sigma_{j=1}^{n} c_{ij} x_{ij}$

  subject to $\Sigma_{j=1}^{n} x_{ij} = 1$ for $i = 1, \cdots, m$

  $\Sigma_{i=1}^{m} x_{ij} \leq m y_j$ for $j = 1, \cdots, n$

  $x_{ij} \geq 0$ for $i = 1, \cdots, m, j = 1, \cdots, n$

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$x_{ij} \leq y_j$ for all i, j	$\Sigma_{i=1}^{m} x_{ij} \leq m y_j$ for all j
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LP2:

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$x_{1j} \leq y_j$

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…

$x_{mj} \leq y_j$

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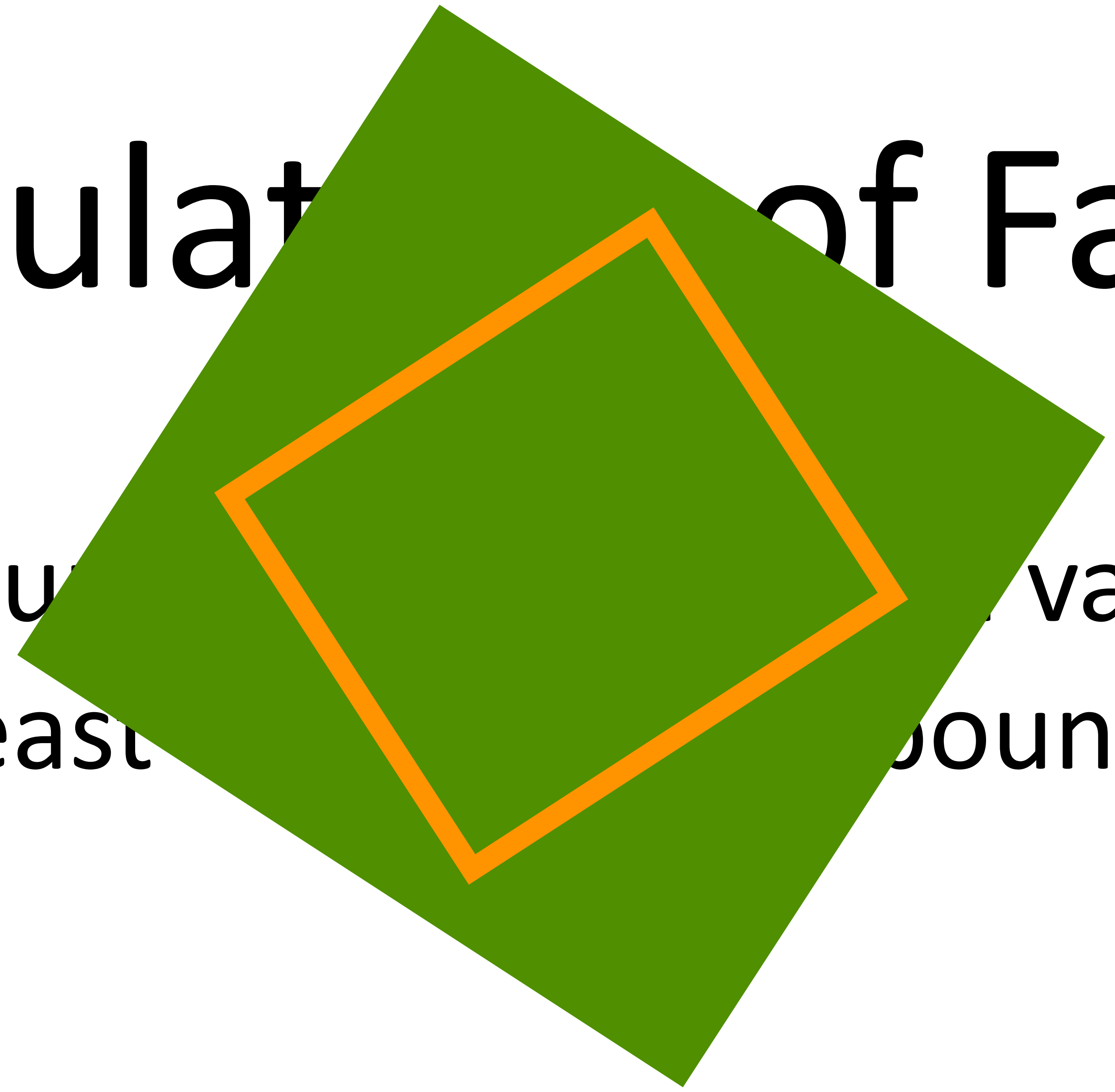
Different formulat[illegible] of Facility Location



- Theorem: The lower bou[illegible] value obtained from the LP-relaxation of LP1 is at least [illegible] bound of the LP-relaxation of LP2

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LP2:

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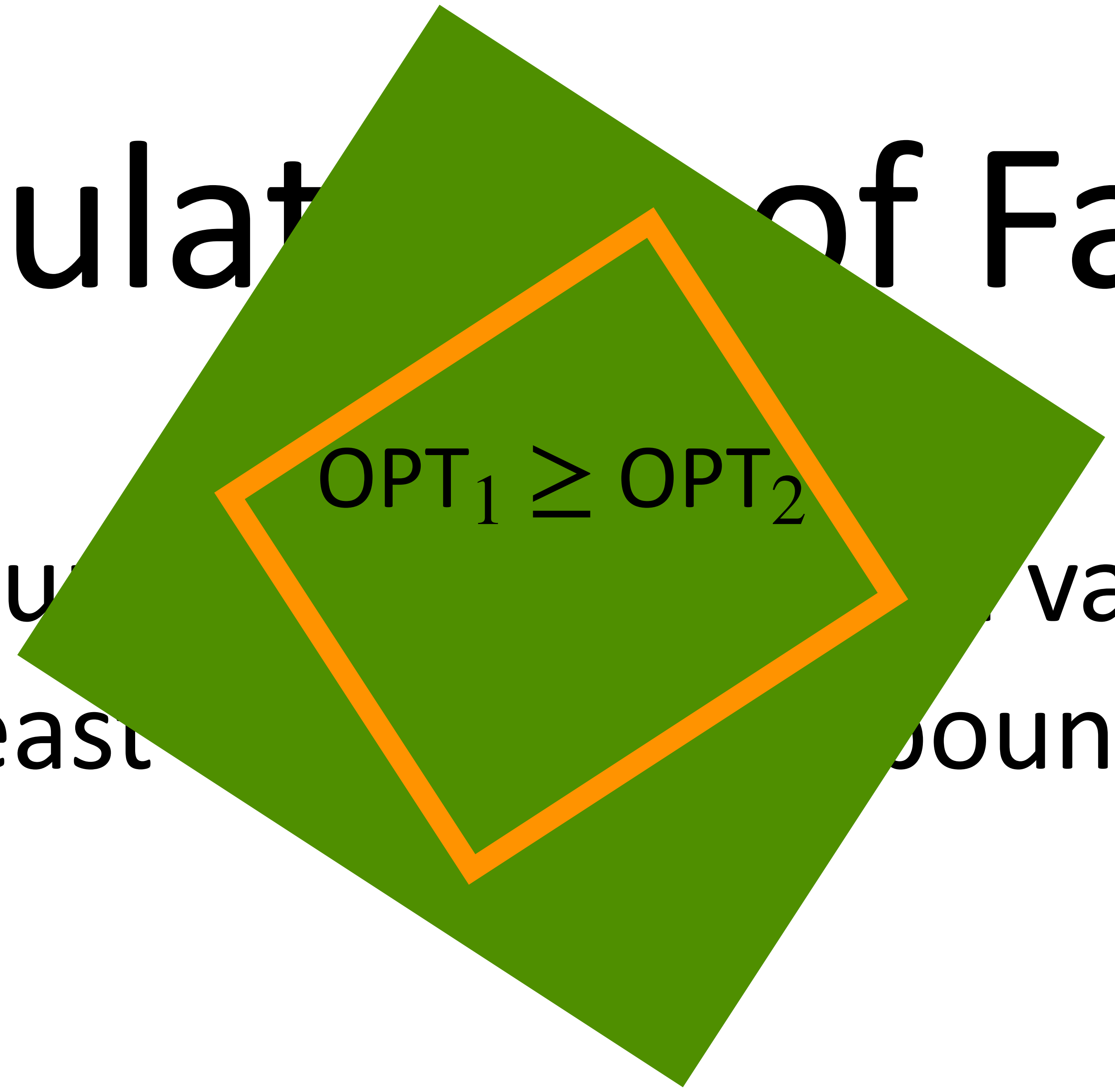
Different formulat of Facility Location



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$\mathrm{OPT}_1 \geq \mathrm{OPT}_2$

integral **OPT** $\geq \mathrm{OPT}_i$

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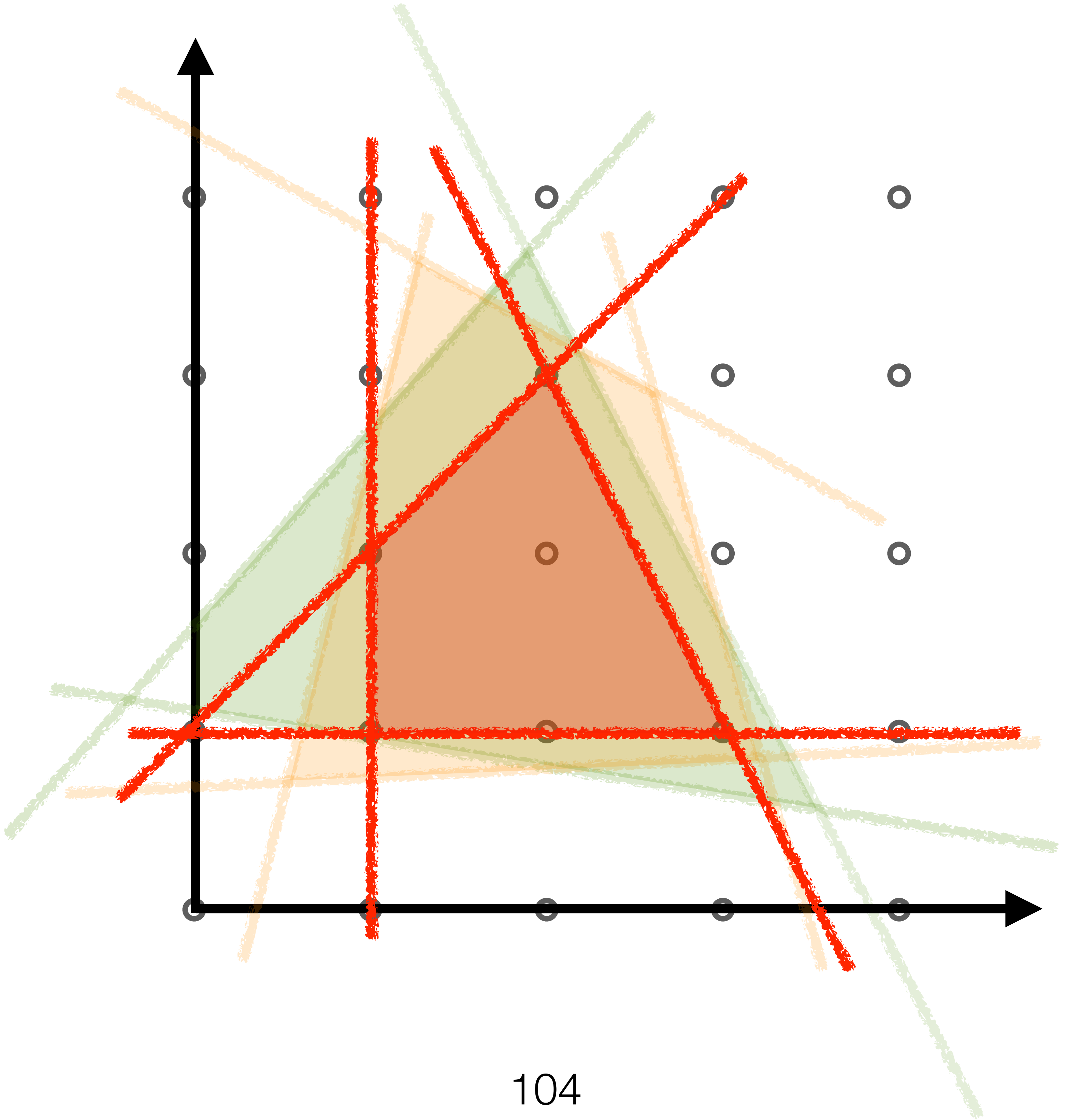
Different formulations of ILP

- Geometrically, we can see that there must be an infinite number of formulations
    - How can we choose between them?

Formulation 1

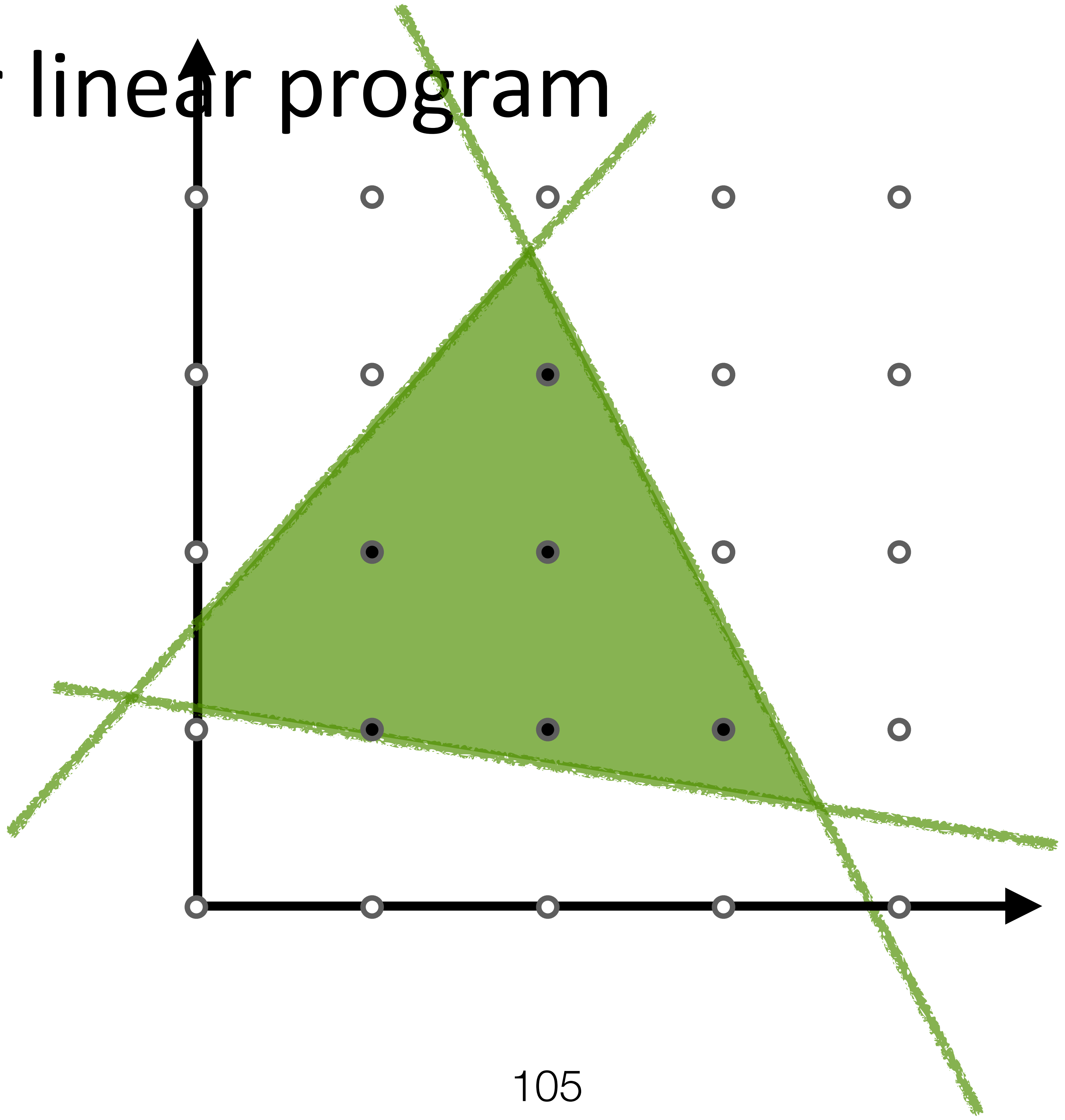
Formulation 2

Ideal formulation



Cutting Plane

- Sometimes, by adding constraints, the integer linear program might be more effective to solve
  - These added constraints should not rule out any feasible solutions to the original integer linear program



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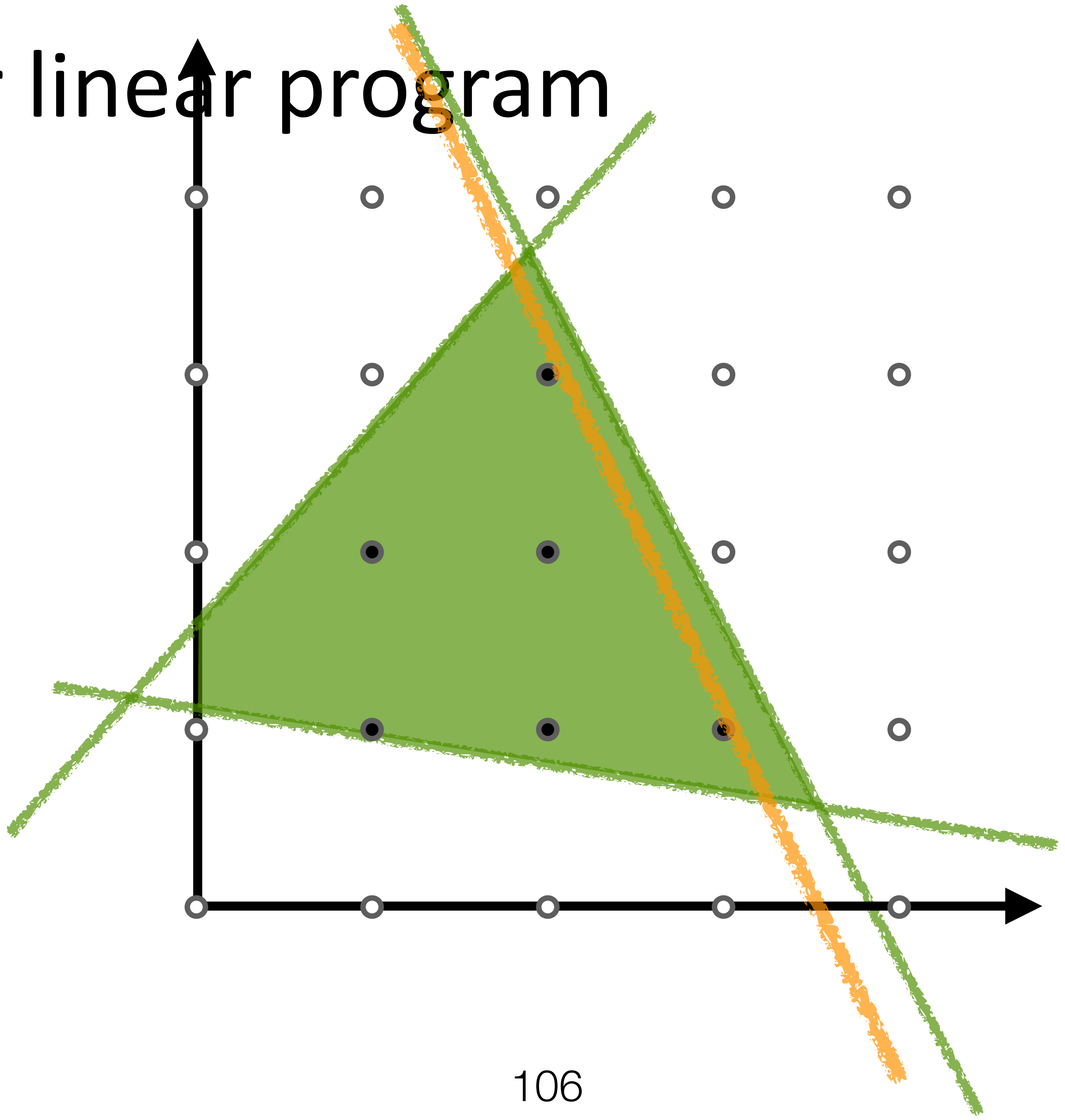

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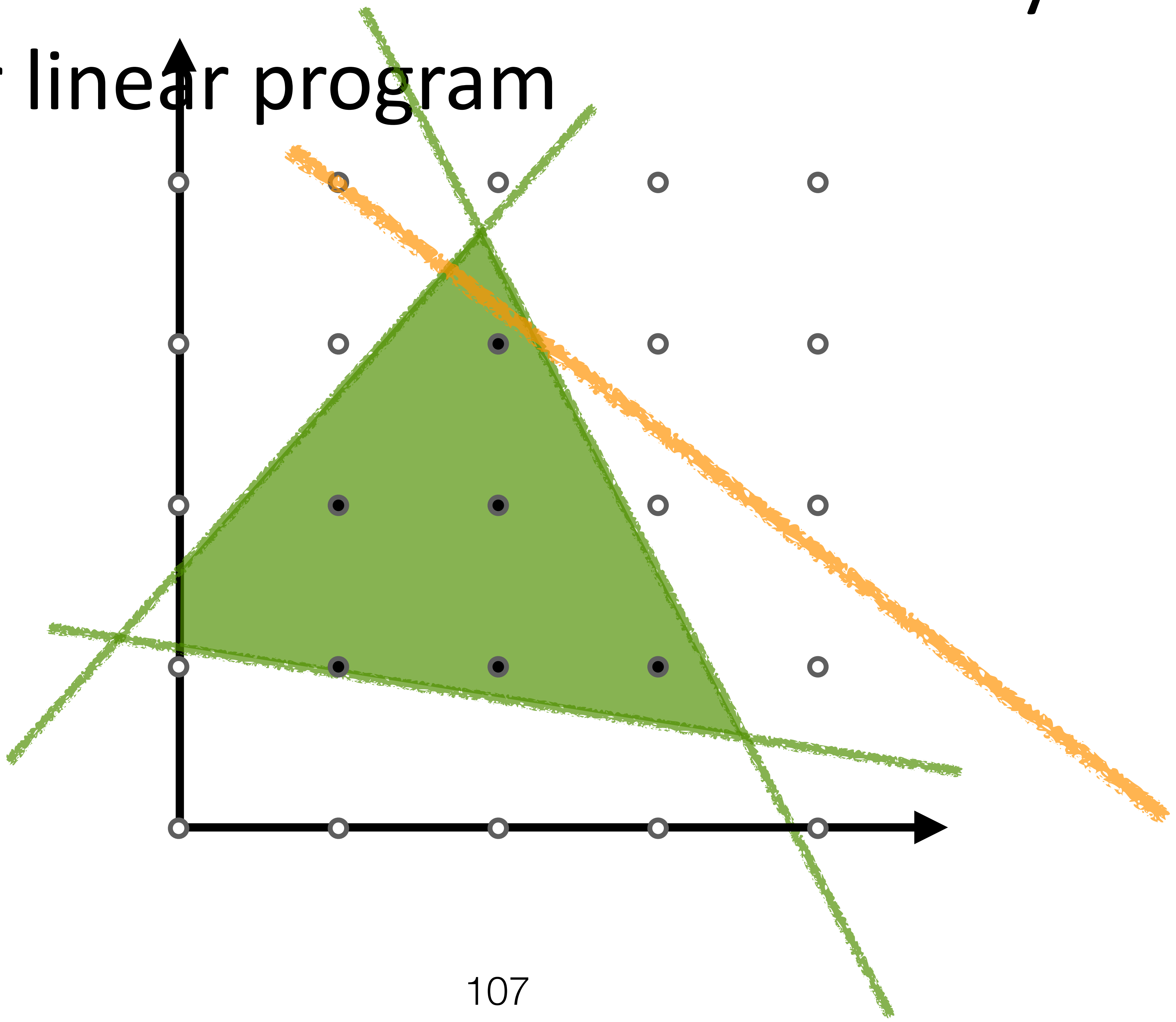

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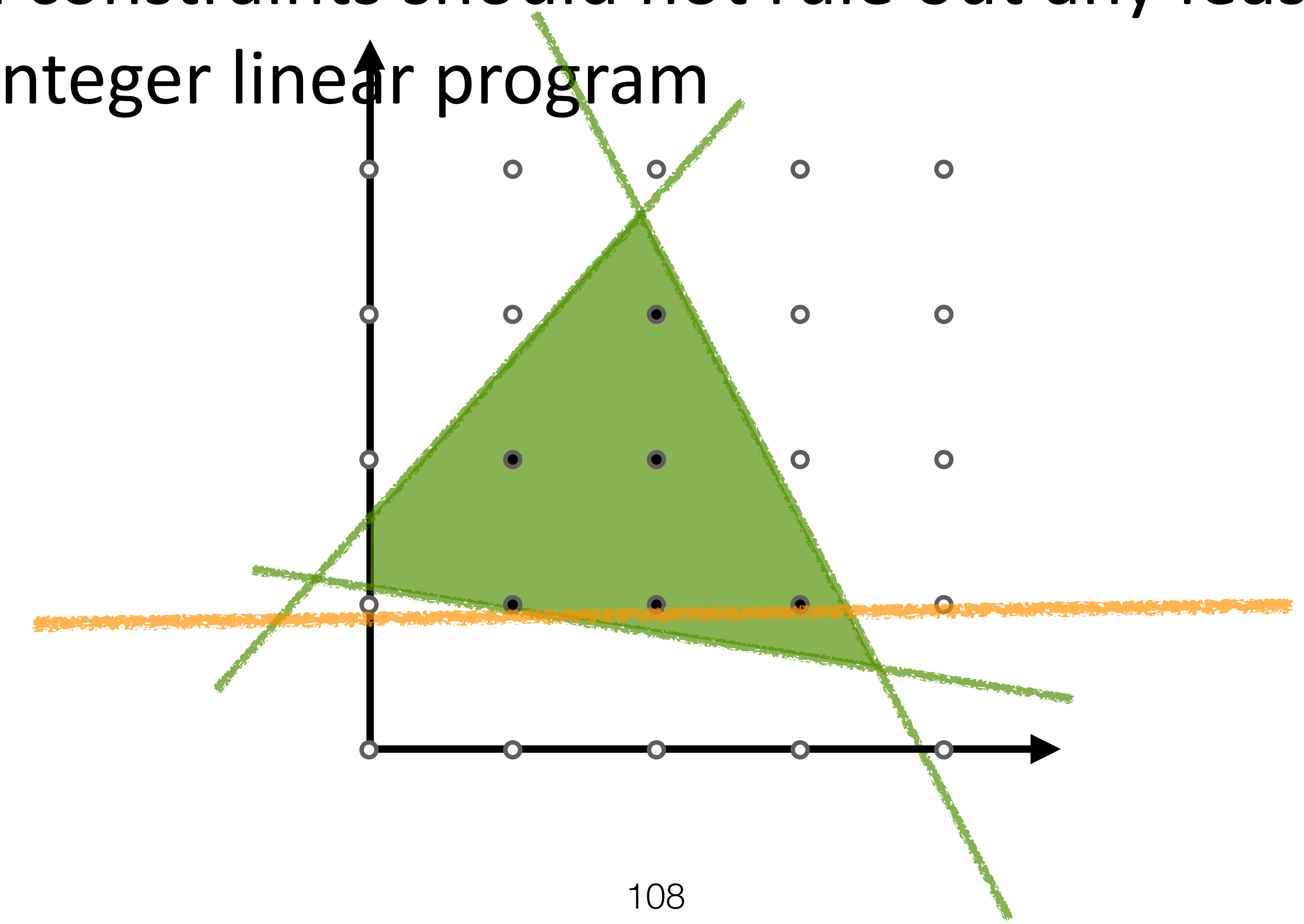

Example 1: adding constraints

Minimize $x_1 + x_2 + x_3 + x_4 + x_5$

s. t. $3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leq -2$

$x_i \in \{0,1\}$ for all i

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- If $x_2 = x_4 = 0$:

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$x_2 + x_4 \geq 1$

$x_1 \leq x_2$

$x_i \in \{0,1\}$ for all i

Example 3: adding constraints

Minimize $\sum_{i \in M, j \in N} c_{ij} x_{ij}$

s. t. $\sum_{i \in M} x_{ij} \leq b_j y_j$ for $j \in N$

$\sum_{j \in N} x_{ij} = a_i$ for $i \in M$

$x_{ij} \geq 0$ and $y_j \in \{0,1\}$

- All feasible solutions satisfy:
  - $x_{ij} \leq b_j y_j$
  - $x_{ij} \leq a_i$

  with $y_j \in \{0,1\}$

  $\Rightarrow x_{ij} \leq \min\{a_i, b_j\} \cdot y_j$

Example 4: adding constraints

Minimize $x_1 + x_2 + x_3 + x_4$

s. t. $13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72$

$x_i \in \mathbb{N}$ for all i

- Divide both sides of the constraint by 11:

$$\frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11}$$

  - Since $x_i \in \mathbb{N}$, $2x_1 + 2x_2 + x_3 + x_4 \geq \frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11} = 6.\cdots$
  - Since $x_i \in \mathbb{N}$, $2x_1 + 2x_2 + x_3 + x_4 \geq 7$

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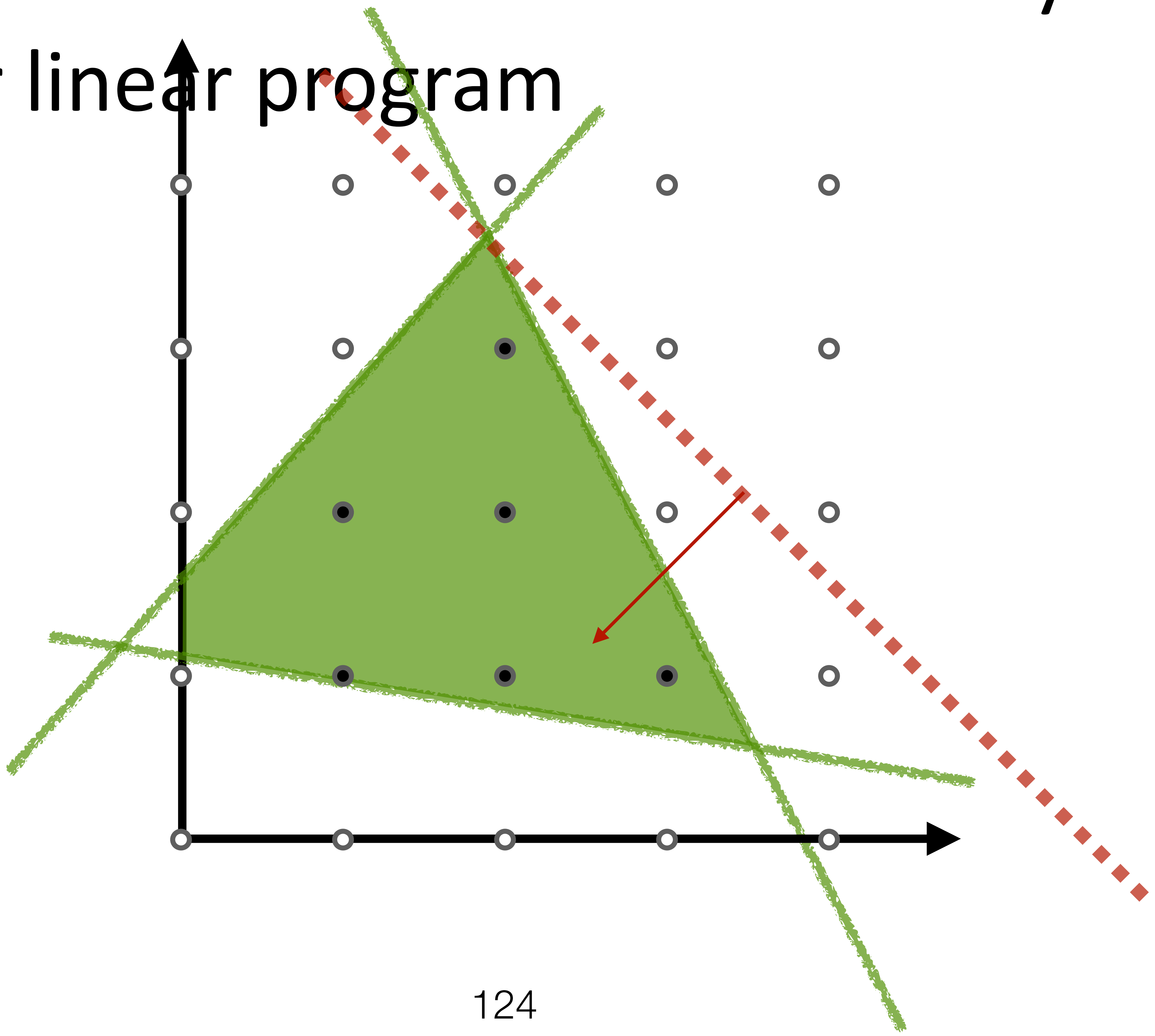

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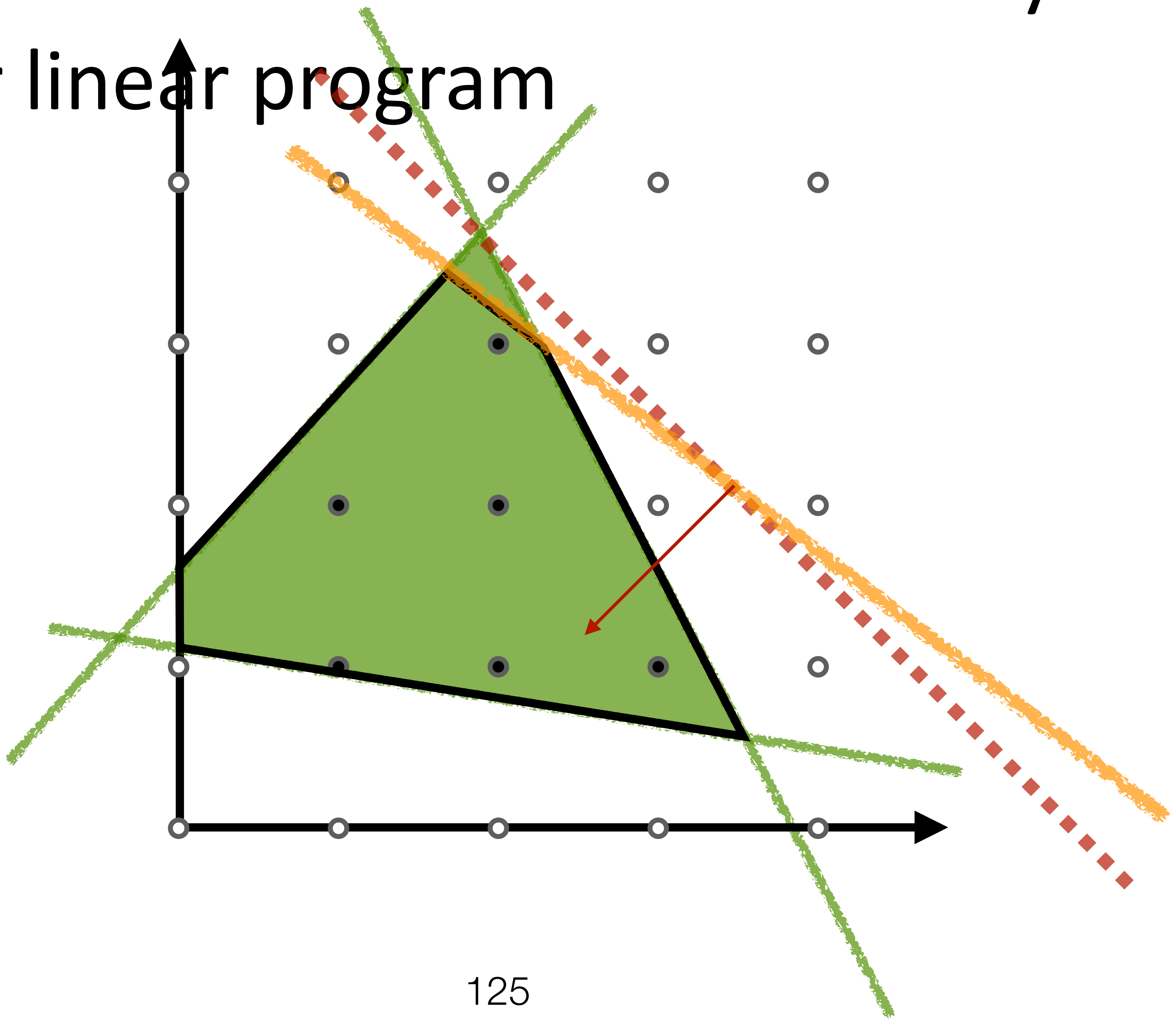

Cutting Plane

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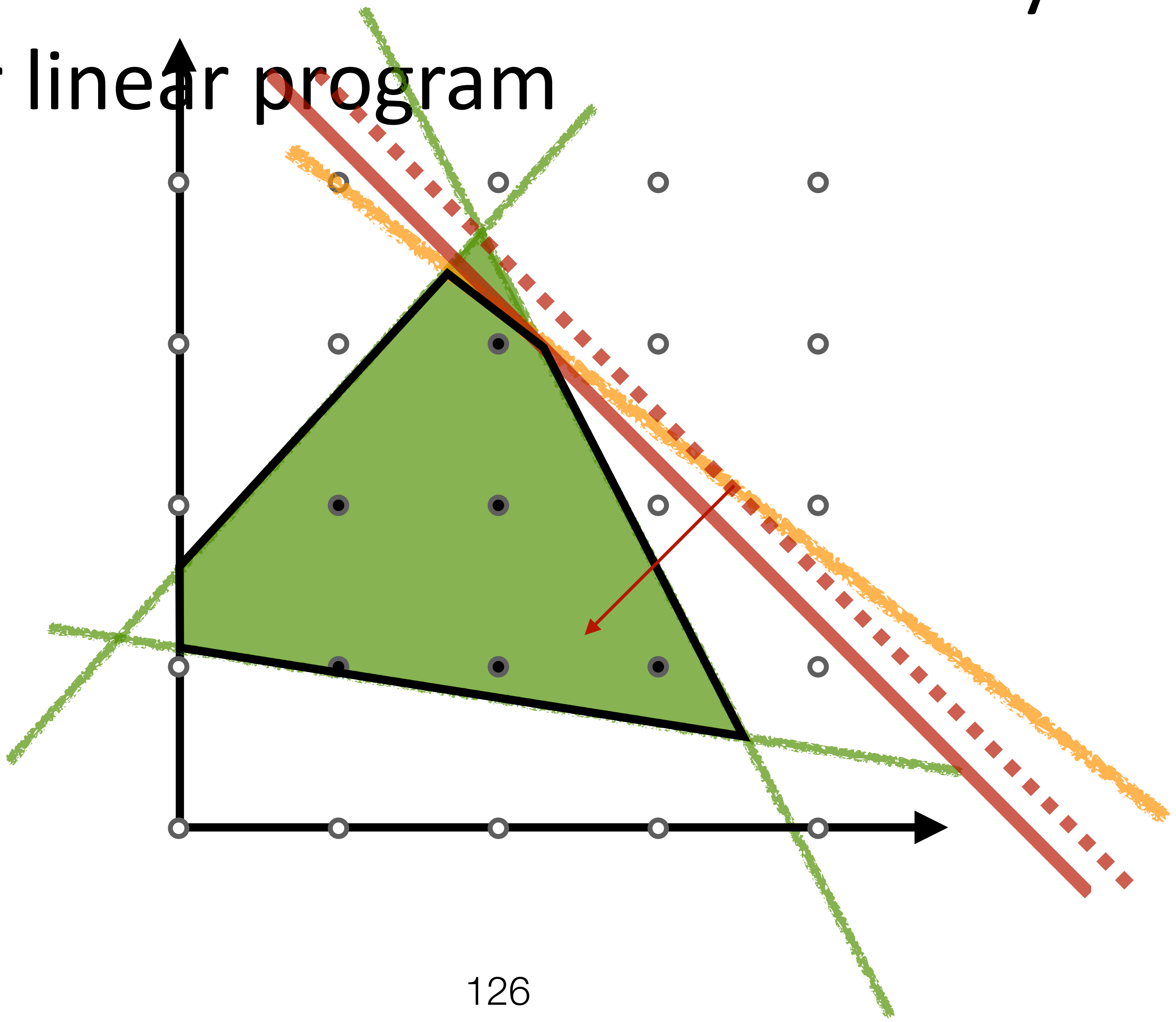

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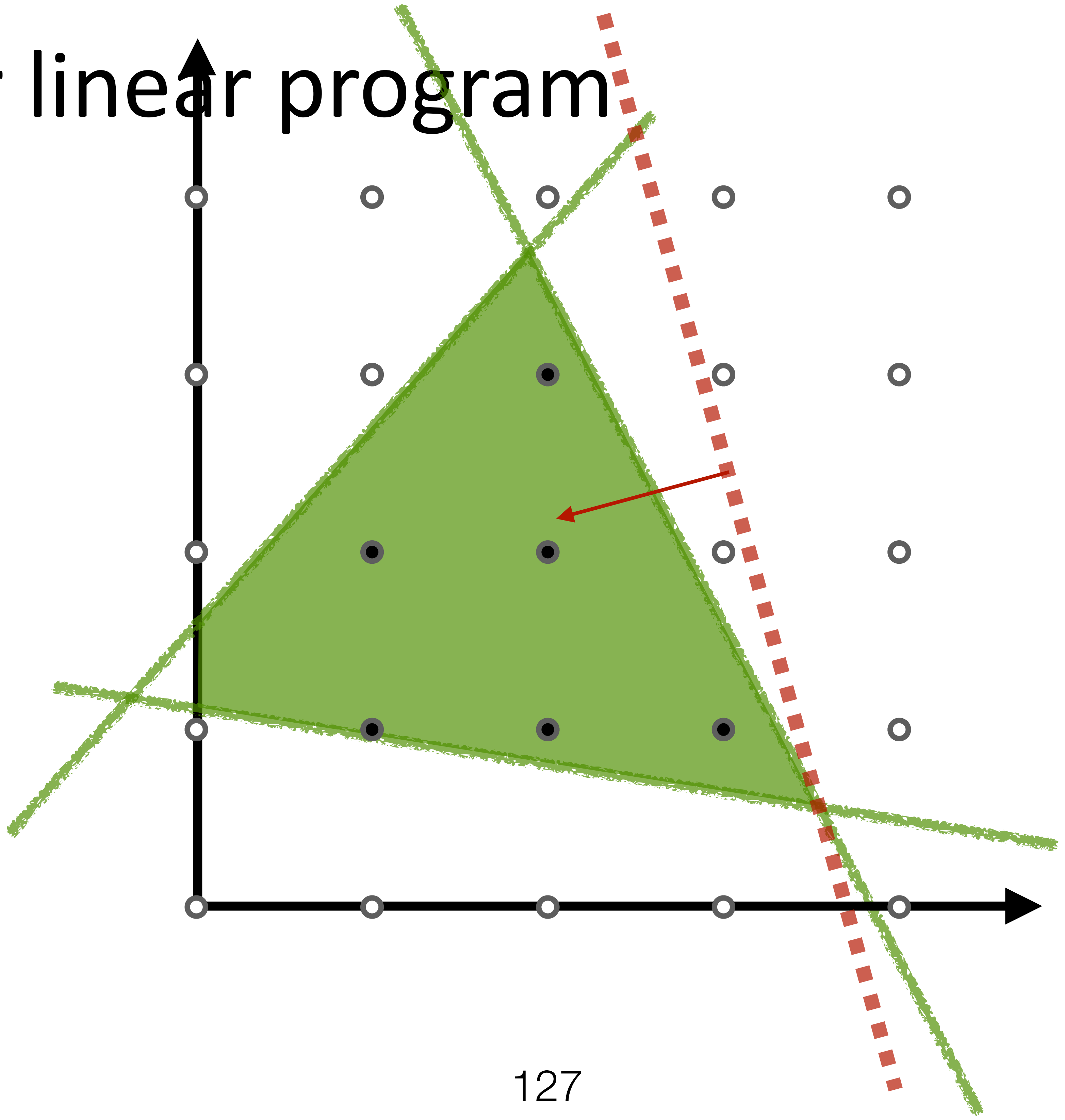

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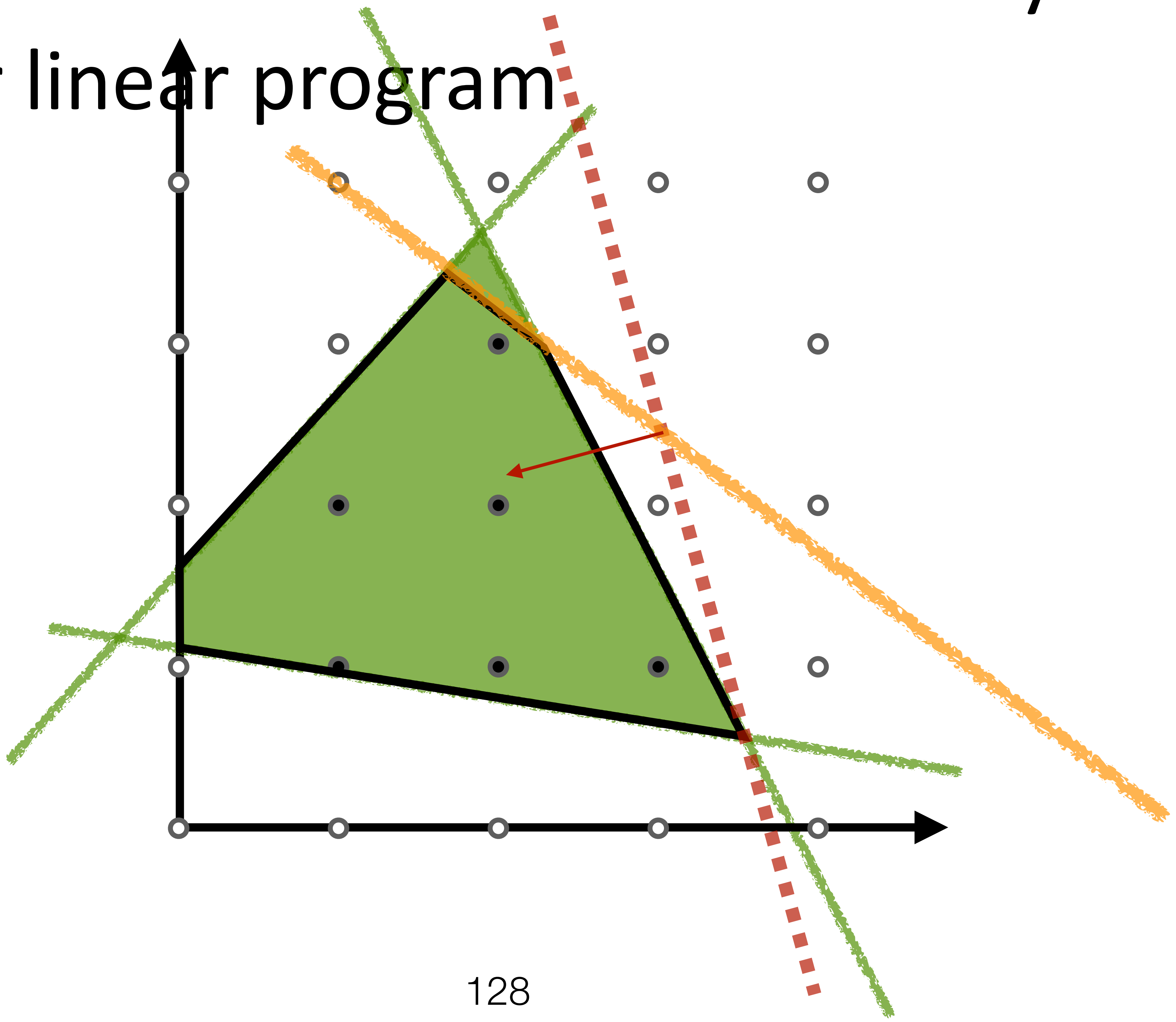

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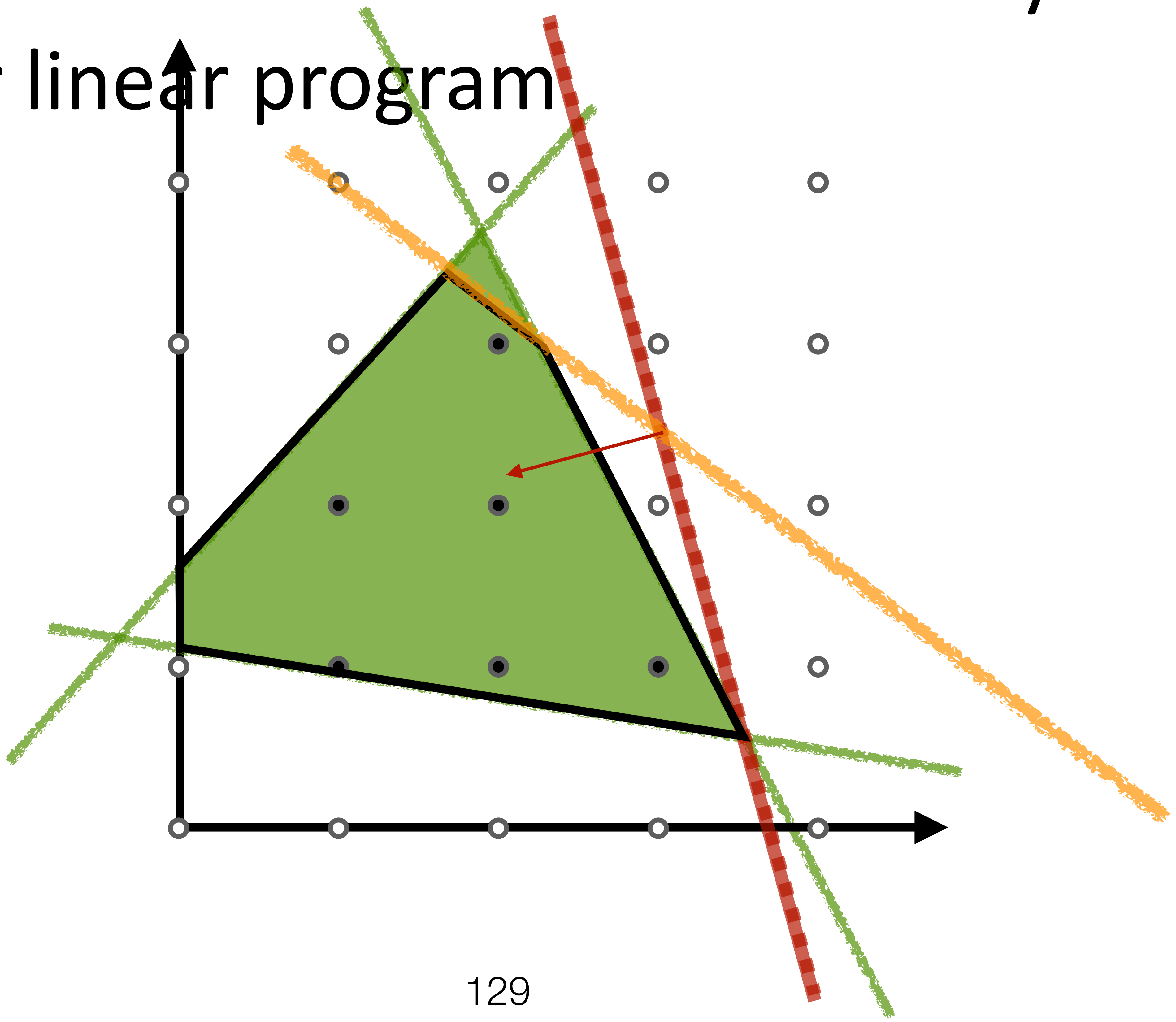

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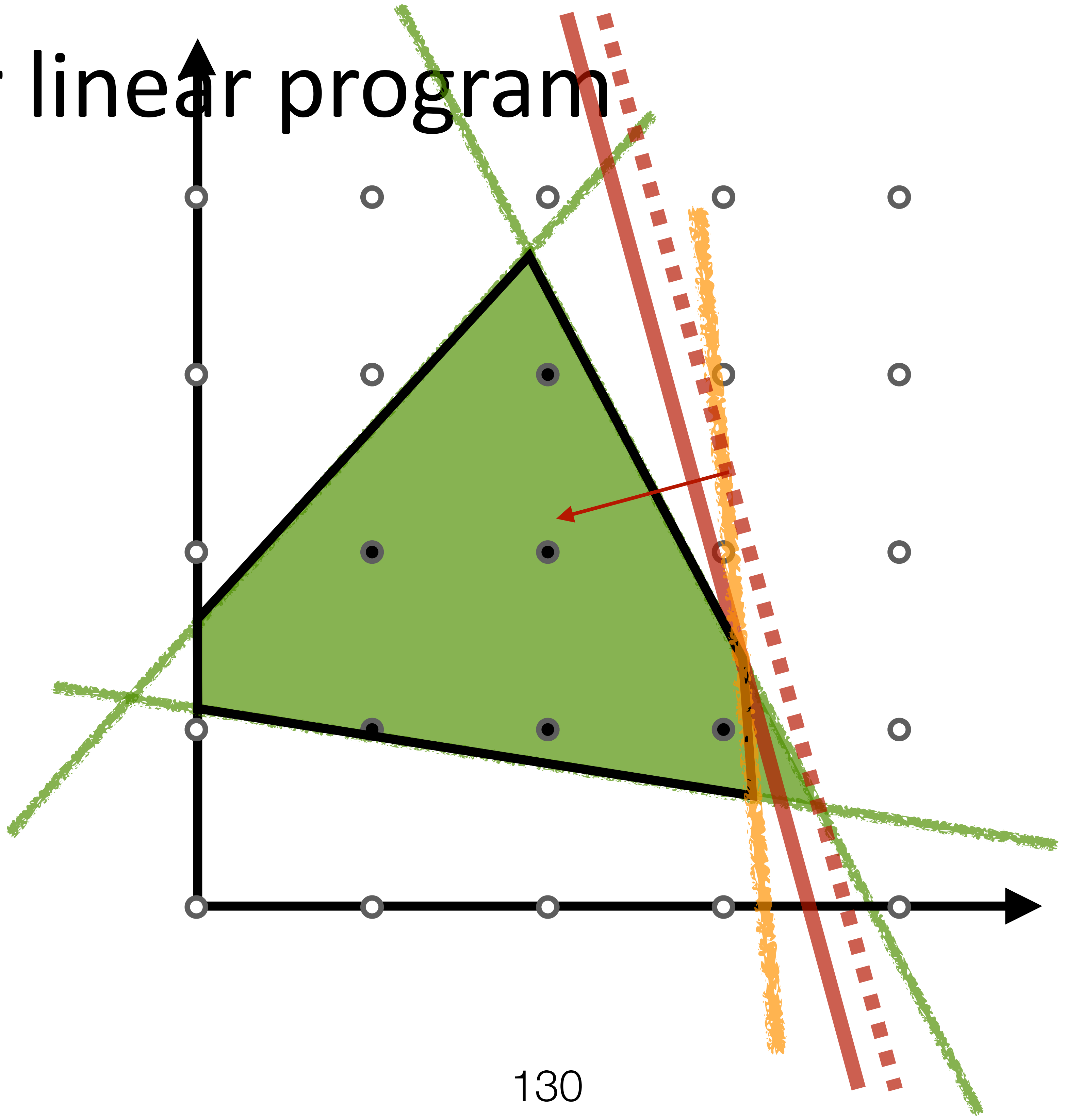

Cutting Planes

- By adding constraints, the solution to relaxed LP might be closer to the solution to the ILP
  - Need to make sure that no feasible integral solution is ruled out by the new constraints

Or constraints

- Recall that in a linear program, every constraint should be satisfied
- What happens if you only need (at least) one of two conditions to be true?

Or constraints

Minimize $\sum_{j \in J} c_j x_j$

s.t. $\sum_{j \in J} a_{1j} x_j \leq b_1$ (1)

$\sum_{j \in J} a_{2j} x_j \leq b_2$ (2)

$x_j \geq 0$ for all j

- where at least one of (1) and (2) is true

Minimize $\sum_{j \in J} c_j x_j$

s.t. $\sum_{j \in J} a_{1j} x_j \leq b_1 + M_1 \cdot y$ (1*)

$\sum_{j \in J} a_{2j} x_j \leq b_2 + M_2 \cdot (1 - y)$ (2*)

$x_j \geq 0$ for all j

$y \in \{0,1\}$

- Introduce $y \in \{0,1\}$ and large enough M_1 and M_2 to indicate if one condition is true

Or constraints

Minimize $\sum_{j \in J} c_j x_j$

s.t. $\sum_{j \in J} a_{1j} x_j \leq b_1$ (1)

$\sum_{j \in J} a_{2j} x_j \leq b_2$ (2)

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$x_j \geq 0$ for all j

$y \in \{0,1\}$

- Introduce $y \in \{0,1\}$ and large enough M_1 and M_2 to indicate if one condition is true
  - If $y = 0$, $(1^*) = (1)$, and (2^*) is more relaxed than $(2) \Rightarrow$ a solution must satisfy (1) but may not satisfy (2)

Or constraints

Minimize $\sum_{j \in J} c_j x_j$

s.t. $\sum_{j \in J} a_{1j} x_j \leq b_1$ (1)

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$y \in \{0,1\}$

- Introduce $y \in \{0,1\}$ and large enough M_1 and M_2 to indicate if one condition is true
  - If $y = 0$, $(1^*) = (1)$, and (2^*) is more relaxed than $(2) \Rightarrow$ a solution must satisfy (1) but may not satisfy (2)
  - The case where $y = 1$ is symmetrical

Or constraints

- Use an indicator variable y again
  - But this time, use y to restrict one condition and relax the other one, so it is not necessary that both the conditions are true

Conditional constraints

- Recall that in a linear program, every constraint should be satisfied
- What happens if we need condition (2) also be true if condition (1) is true?

Conditional constraints

$$\sum_{j \in J} a_{1j} x_j \leq b_1 \qquad (1)$$

$$\sum_{j \in J} a_{2j} x_j \leq b_2 \qquad (2)$$

- if condition (1) is satisfied, then (2) must also be satisfied

- If P then $Q \Leftrightarrow$ not P or Q

P	Q	If P then Q	not P	not P or Q
T	T		F	
T	F		F	
F	T		T	
F	F		T	

Conditional constraints

$$\sum_{j \in J} a_{1j} x_j \leq b_1 \qquad (1)$$

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P	Q	If P then Q	not P	not P or Q
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Conditional constraints

$$\sum_{j \in J} a_{1j} x_j \leq b_1 \qquad (1)$$

Not (1): $\sum_{j \in J} a_{1j} x_j > b_1$

$$\sum_{j \in J} a_{2j} x_j \leq b_2 \qquad (2)$$

- if condition (1) is satisfied, then (2) must also be satisfied

- If P then $Q \Leftrightarrow$ not P or Q

P	Q	If P then Q	not P	not P or Q
T	T	T	F	T
T	F	F	F	F
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Conditional constraints

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Not (1): $\sum_{j \in J} a_{1j} x_j > b_1$

Not (1): $\sum_{j \in J} a_{1j} x_j \geq b_1 + \varepsilon$

- if condition (1) is satisfied, then (2) must also be satisfied

- If P then $Q \Leftrightarrow$ not P or Q

P	Q	If P then Q	not P	not P or Q
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Conditional constraints

$$\sum_{j \in J} a_{1j} x_j \leq b_1 \quad (1)$$

$$\sum_{j \in J} a_{2j} x_j \leq b_2 \quad (2)$$

- if condition (1) is satisfied, then (2) must also be satisfied

Not (1): $\sum_{j \in J} a_{1j} x_j > b_1$

Not (1): $\sum_{j \in J} a_{1j} x_j \geq b_1 + \varepsilon$

Not (1) or (2):

$$\sum_{j \in J} a_{1j} x_j \geq b_1 + \varepsilon - M_1 \cdot y$$

$$\sum_{j \in J} a_{2j} x_j \leq b_2 + M_2 \cdot (1 - y)$$

- If P then $Q \Leftrightarrow$ not P or Q

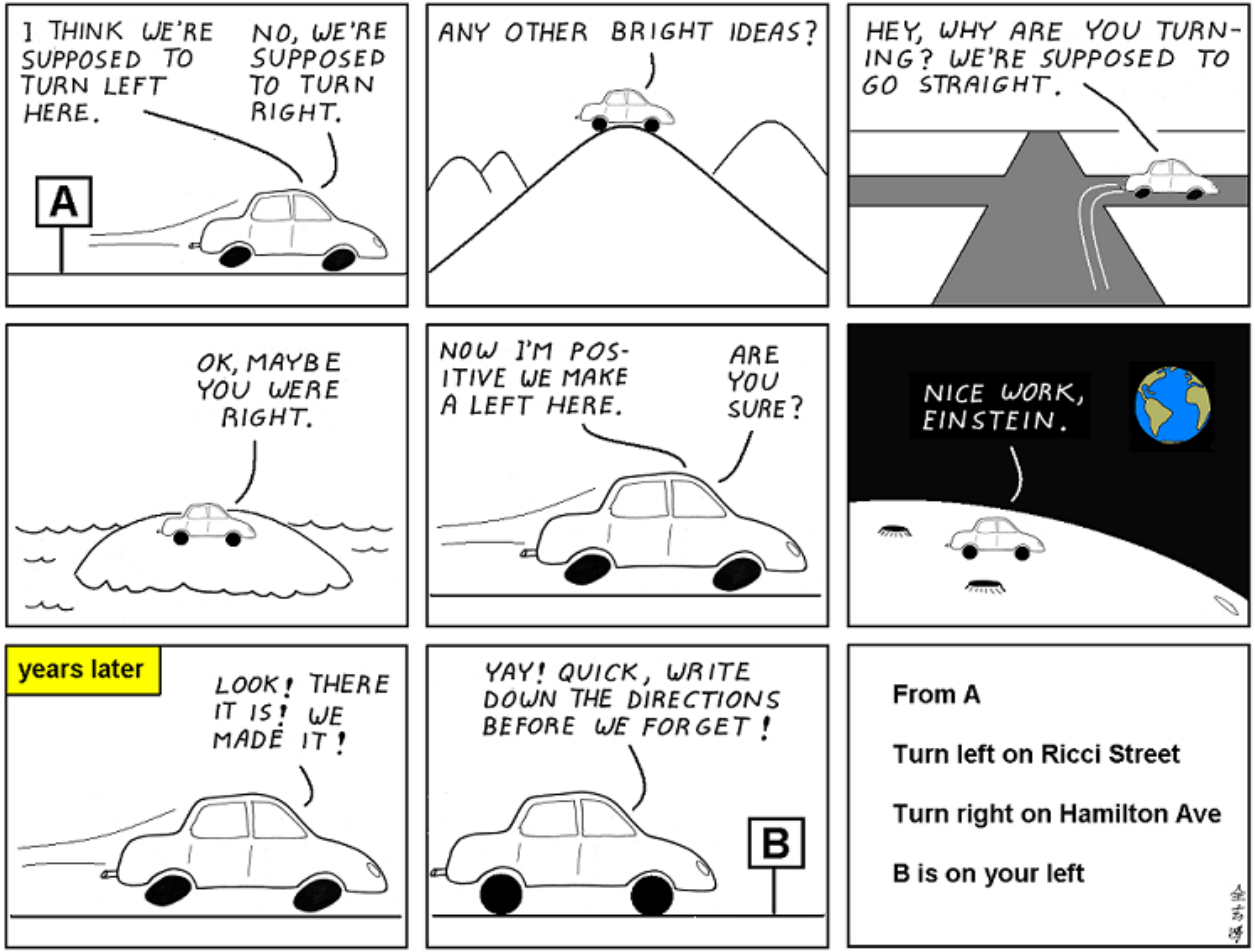
P	Q	If P then Q	not P	not P or Q
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Conditional constraints

- An application for the “or condition” method
  - If P then $Q \Leftrightarrow$ not P or Q
  - Use $+\varepsilon$ where ε is very small to deal with the strict inequality

It’s **obvious**

— by Abstruse Goose



This is how most mathematical proofs are written.