# Exercise: Different formulations and branch-and-bound method

## 1 Set cover

Recall the set cover problem mentioned in the lecture. Here, we formulate the problem in a different way (the difference is in bold):

Let $M = \{1, 2, \cdots, m\}$ be the set of regions, and $N = \{1, 2, \cdots, n\}$ be the set of potential centers. **Let $R_j$ be the regions that can be serviced by a center** $j \in N$, and $c_j$ its installation cost. We want to choose a minimum-cost set of service centers so that each region is covered.

We further define $nm$ variables $a_{ij}$ such that $a_{ij} = 1$ if $i \in R_j$, and $a_{ij} = 0$ otherwise. Use the defined variables to formulate the set cover problem.

We define variable $x_j$ for every center $j$ such that $x_j = 1$ if center $j$ is selected, and $x_j = 0$ otherwise.

$$\begin{aligned}
&\text{minimize} && \sum_{j=1}^{n} c_j x_j && && (1)\\
&\text{subject to} && \sum_{j=1}^{n} a_{ij} x_j \geq && 1 && \text{for } i = 1, 2, \cdots, m && (2)\\
& && x_j \in && \{0, 1\} && \text{for } j = 1, 2, \cdots, n && (3)
\end{aligned}$$

The constraint (2) means that for any region $i$, at least one center that can serve it is selected.

## 2 Single source all destination shortest paths

Consider the single source shortest paths problem defined as follows:

Given a directed graph $G = V, E$, each edge $(u, v) \in E$ has a non-negative length $\ell_{uv}$. We want to find the shortest paths from $s \in V$ to **each of the vertices** with the shortest lengths.

We define the variable $d_v$ for every vertex $v \in V$, which is the length of the shortest path from $s$ to $v$. It is known that for all $v$, $d_v \leq \min_{u:(u,v) \in E}\{d_u + \ell_{uv}\}$. Use the variables to formulate the single source shortest paths problem.

$$\begin{aligned}
&\text{minimize} && \sum_{v \in V} d_v && && (4)\\
&\text{subject to} && d_u + \ell_{uv} \geq && d_v && \text{for all } (u, v) \in E && (5)\\
& && d_v \geq && 0 && \text{for all } v \in V && (6)
\end{aligned}$$

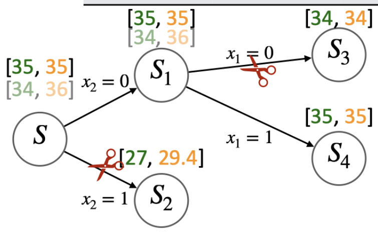
The constraint (5) comes from the fact that $d_v \leq \min_{u:(u,v) \in E}\{d_u + \ell_{uv}\}$. It follows that $d_v \leq d_u + \ell_{uv}$ for every $u$ and $v$ where $(u, v)$ edge exists.

## 3 Branch-and-Bound algorithm

1. Recall the example of running the Branch-and-Bound algorithm on the knapsack problem in the lecture. Run the algorithm on the same example again but with order item 2, item 1, item 3, item 5, and then item 4.

# Use Branch-and-Bound to solve Knapsack

| Item | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| return | 8 | 12 | 7 | 15 | 12 |
| outlay **15** | 4 | 8 | 3 | 6 | 5 |
| outlay/return | 2 | 1.5 | 2.333 | 2.5 | 2.4 |




162

2. Consider the following knapsack problem: There are 4 items, $x_1$, $x_2$, $x_3$, and $x_4$. The $(return, outlay)$ pairs of the items are $(17, 5)$, $(10, 3)$, $(25, 8)$, and $(17, 7)$, respectively. Solve the knapsack problem by running then Branch-and-Bound algorithm.