Algorithms for Decision Support

Introduction to Algorithmic Game Theory

Outline

- Fundamental concepts
  - Game, players, strategies, payoffs/costs
- Nash Equilibrium
- Price of Anarchy
  - Selfish load balancing
- Mechanism design
  - Auction
  - Vickrey-Clarke-Groves mechanism

Prisoner’s Dilemma

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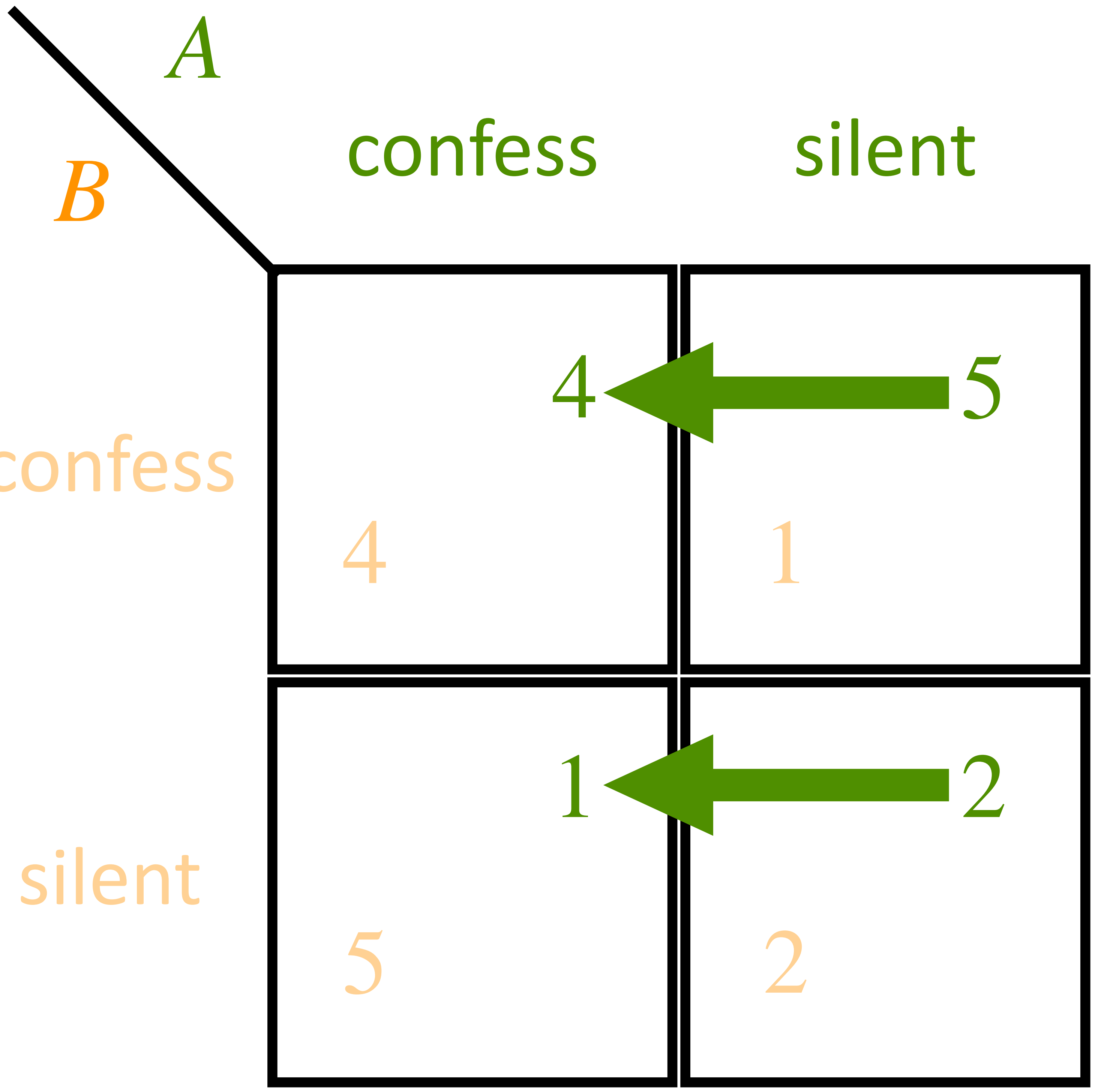

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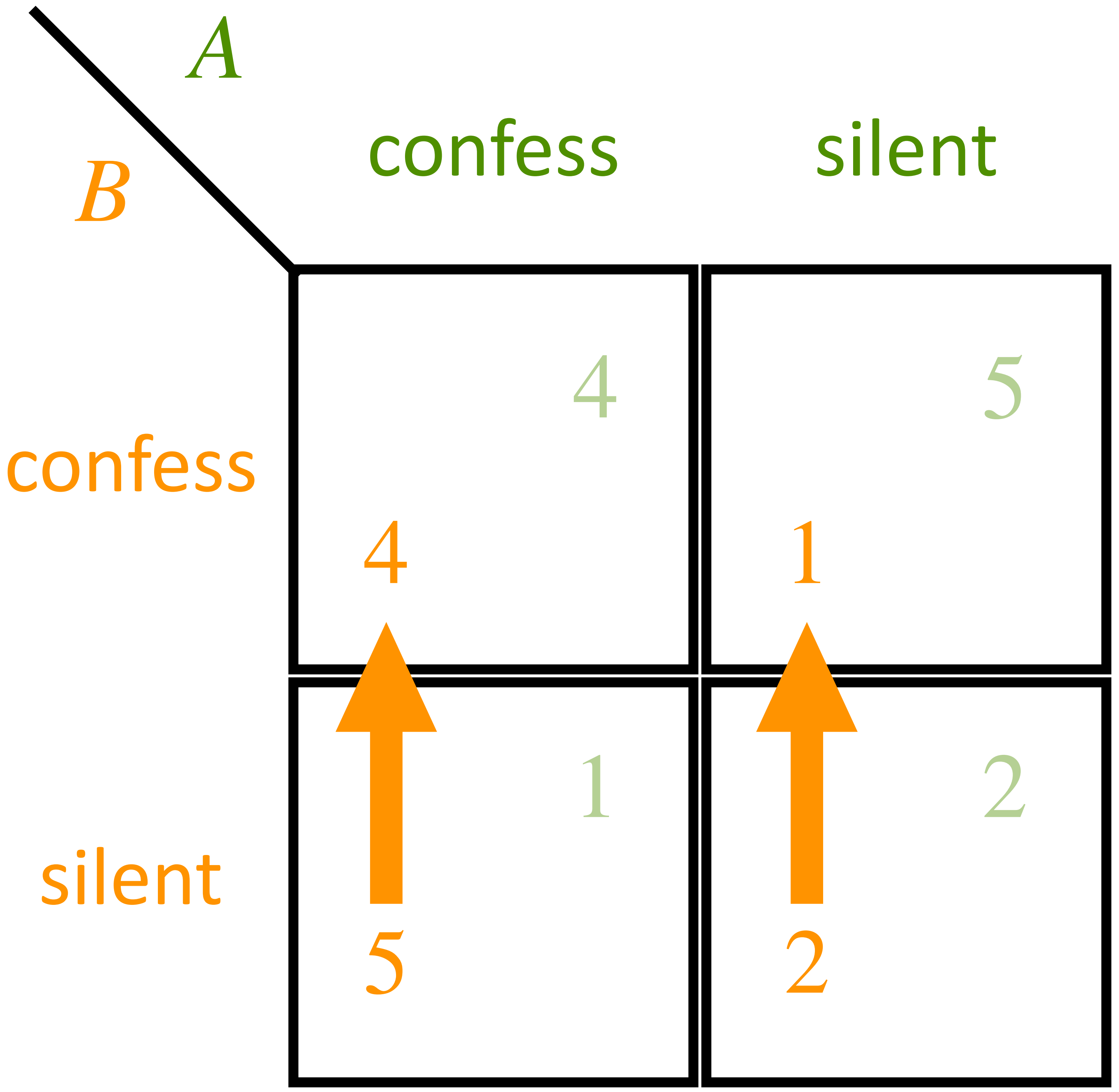

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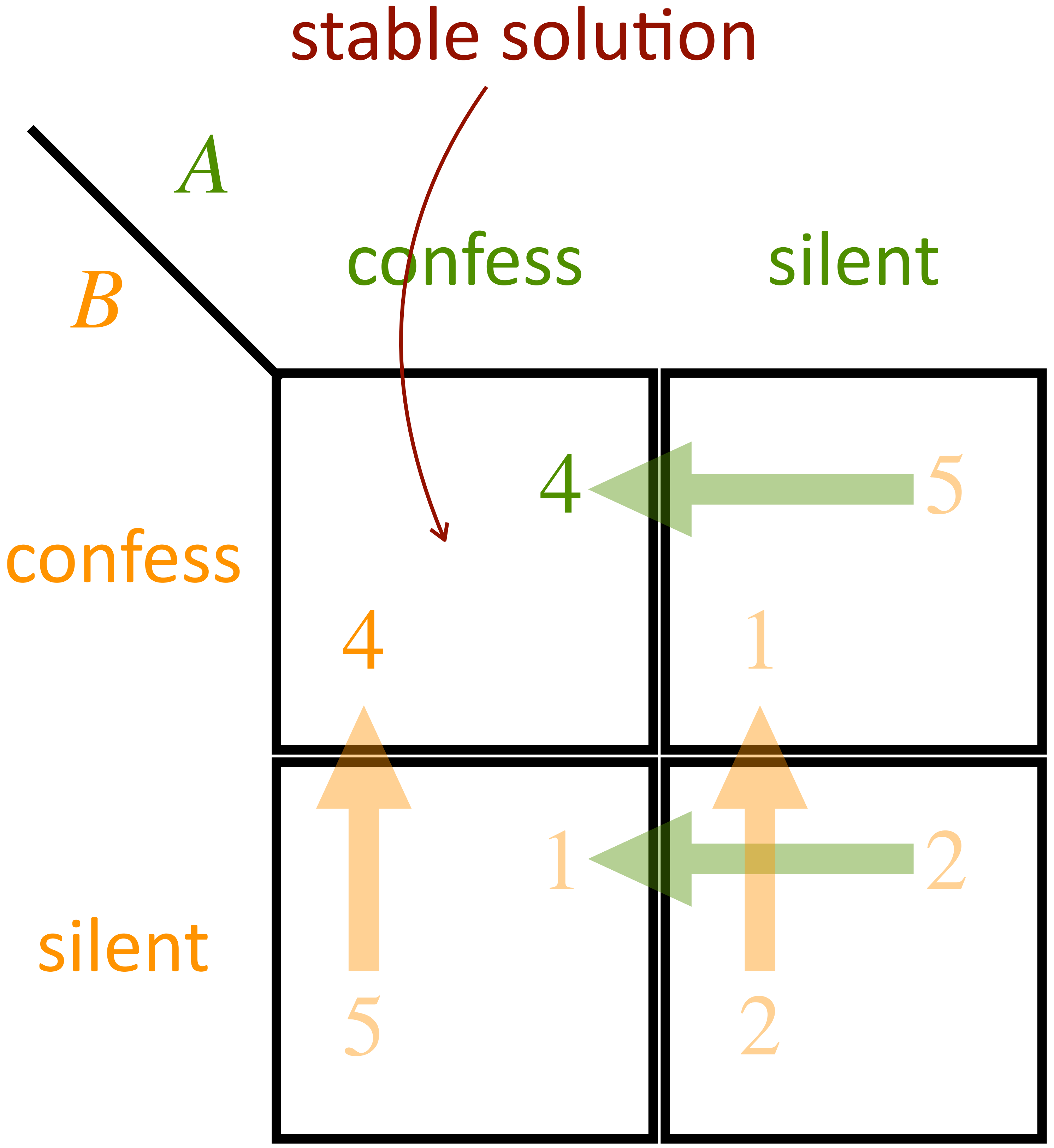

stable solution

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B

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Evening Together

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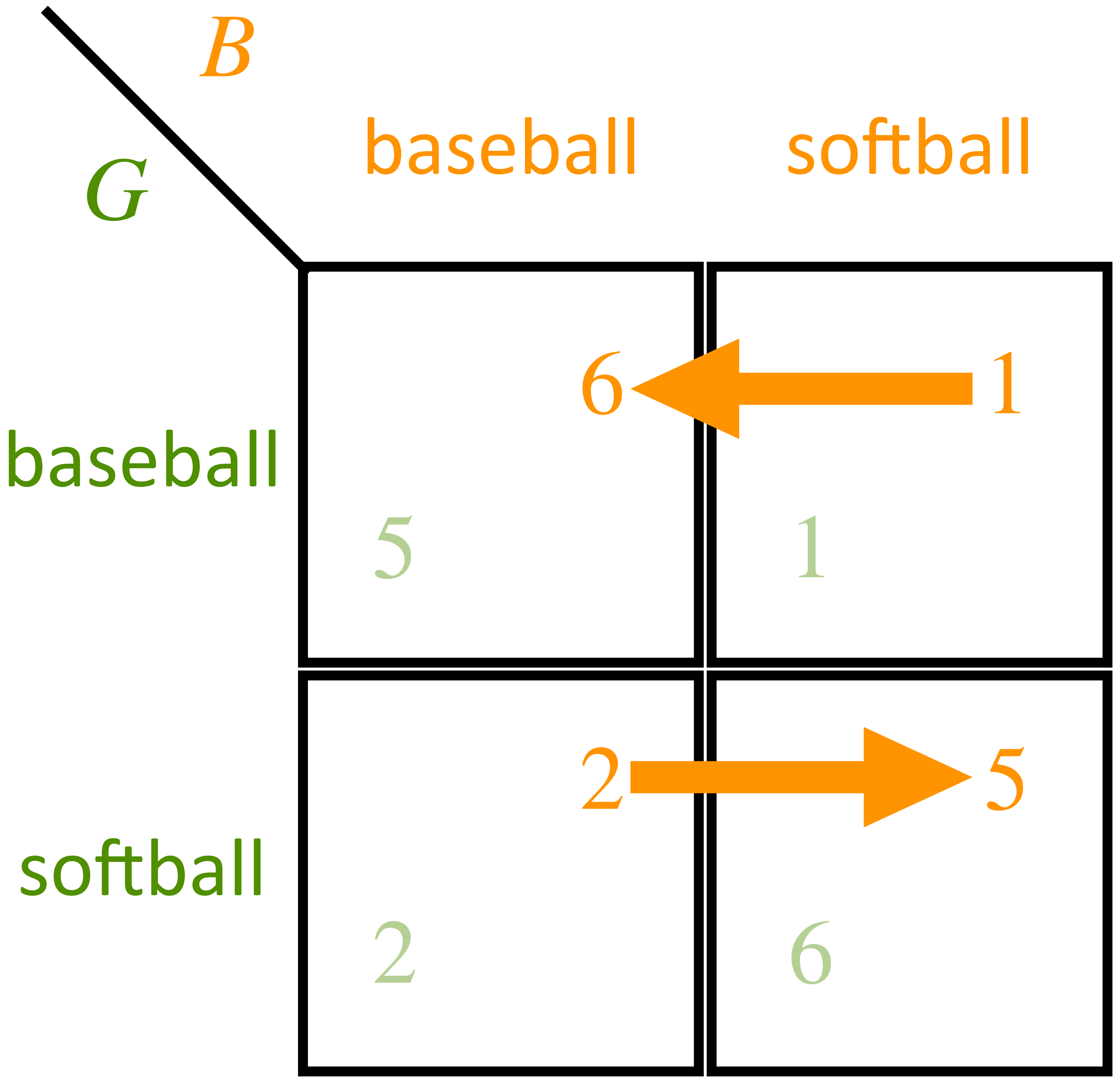

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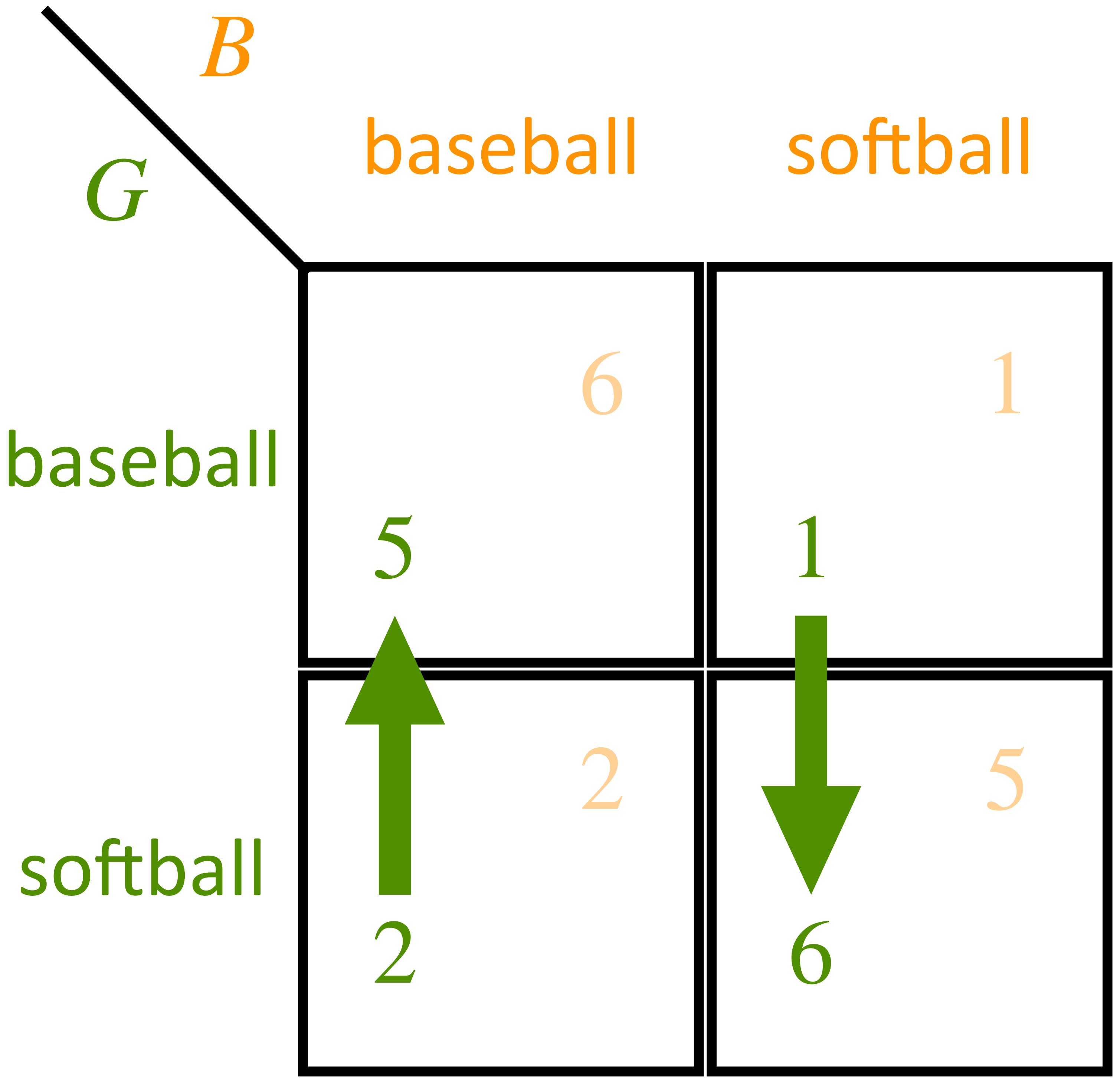

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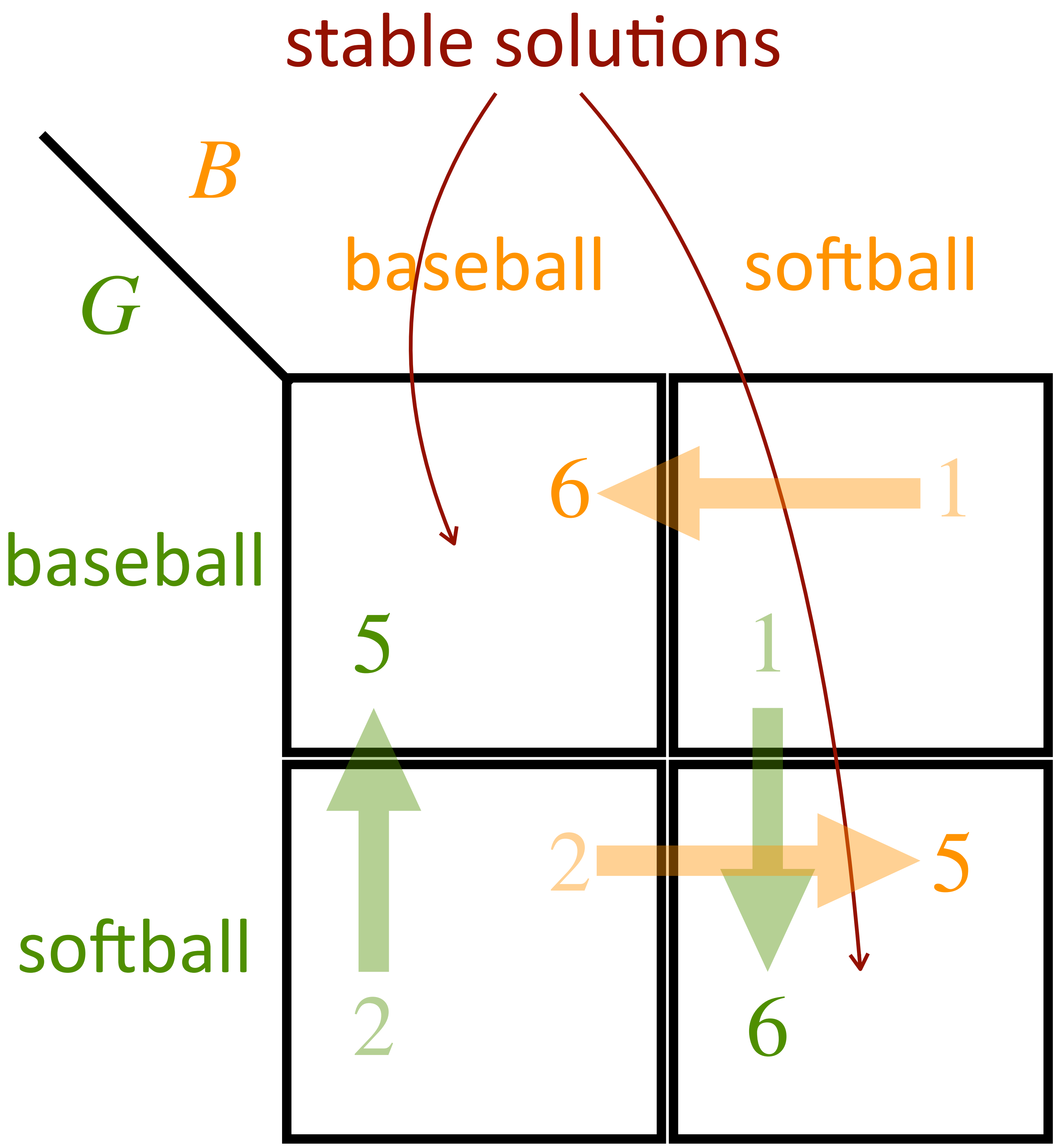

stable solutions

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Matching Pennies

Matching Pennies

- Two players, each having a penny
  - Two strategies: head (H) or tail (T)
    - The row player wins if the two pennies match
    - The column player wins if the two pennies do not match

R \ C	head	tail
head	-1 1	1 -1
tail	1 -1	-1 1

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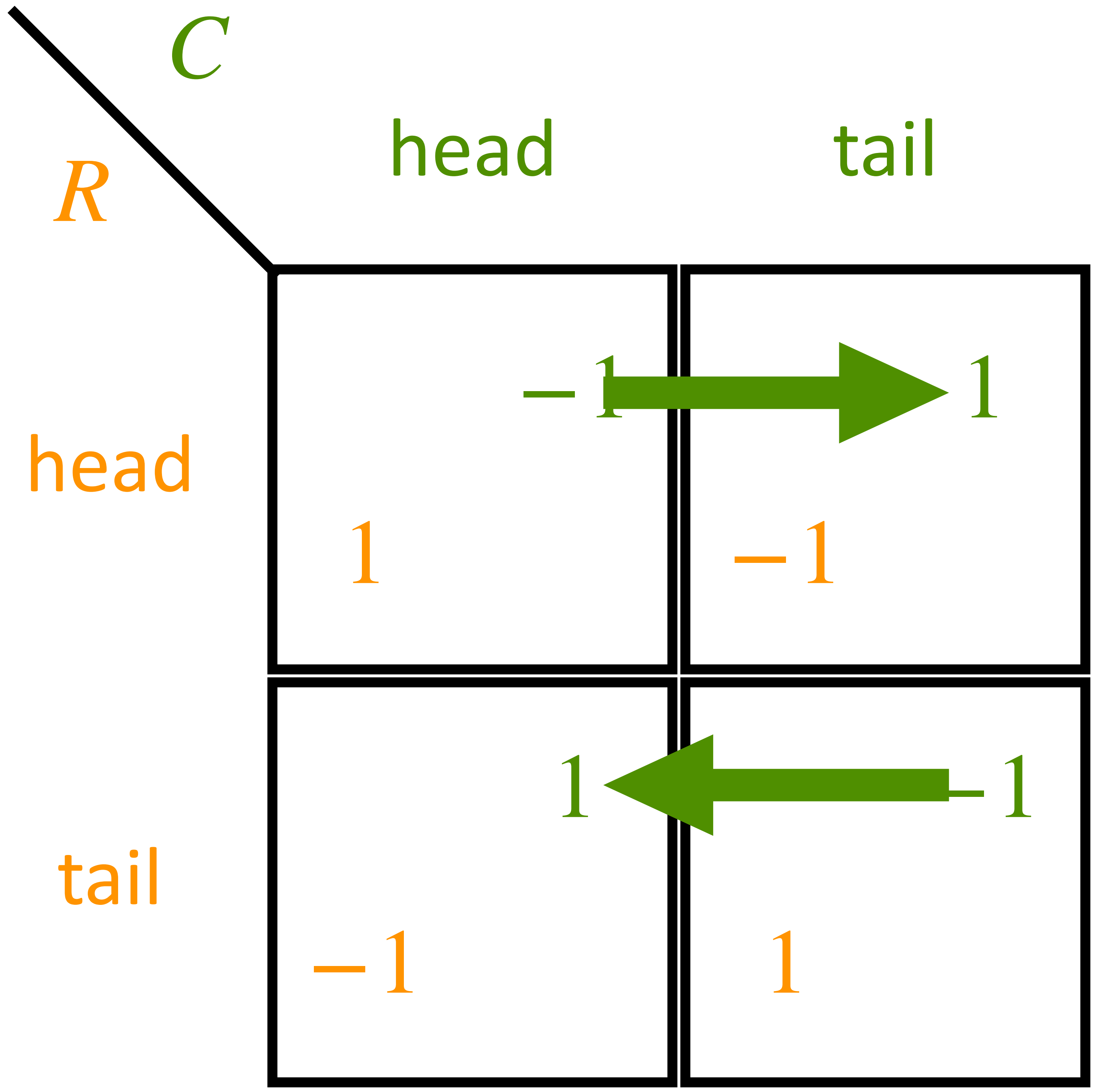

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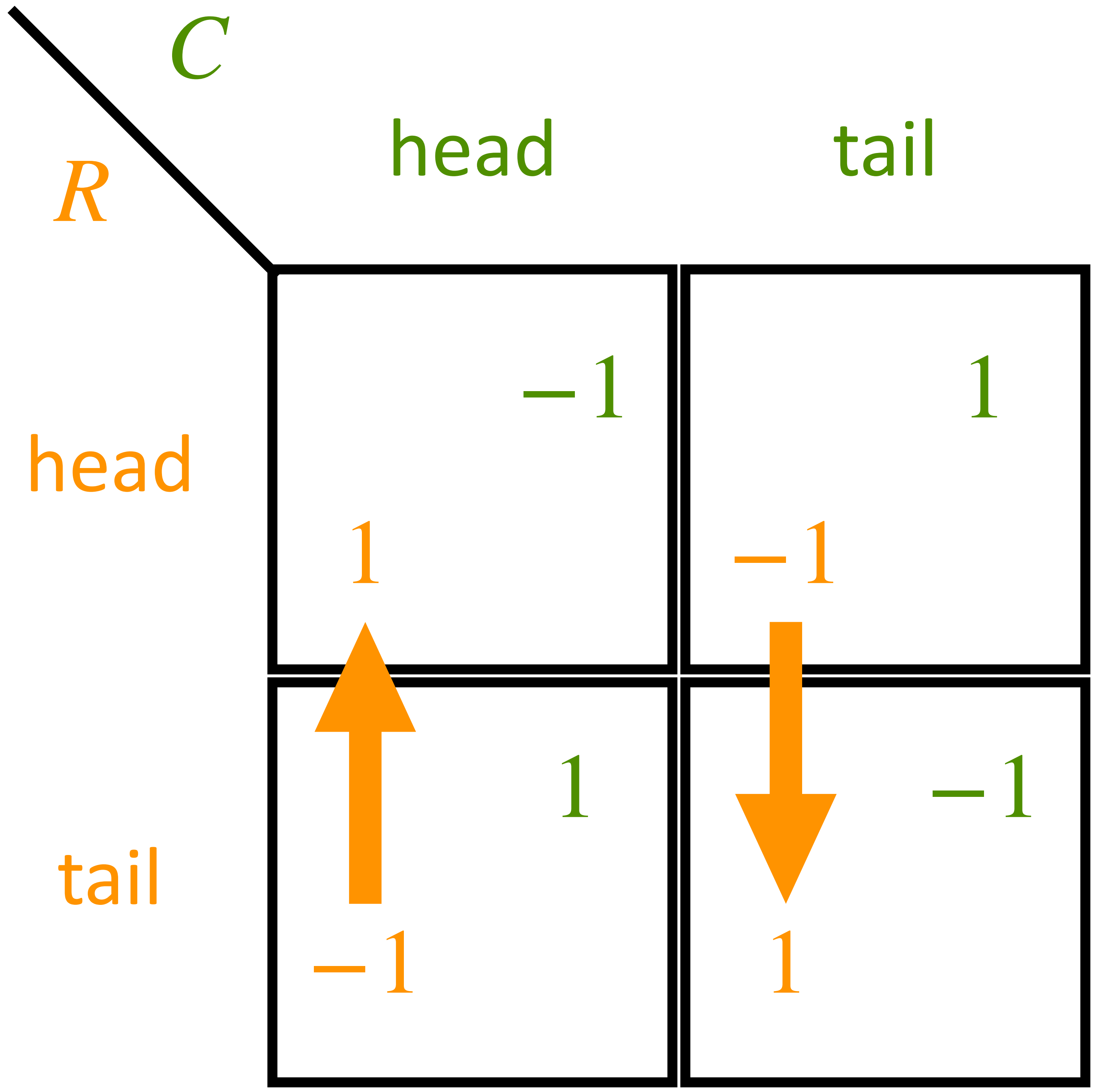

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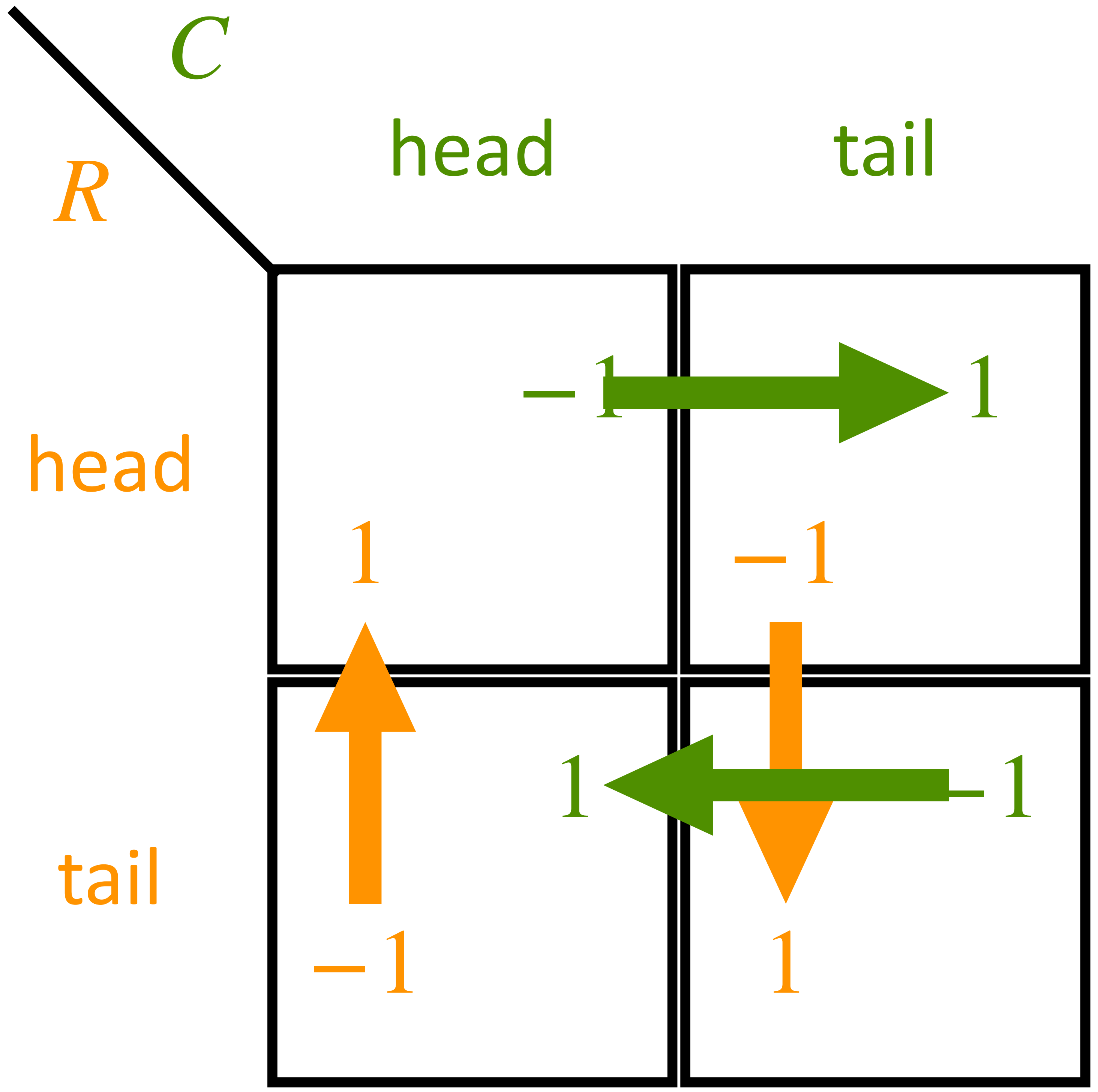
No stable solution!



Tragedy of Commons

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- n players want to have a part of a shared channel
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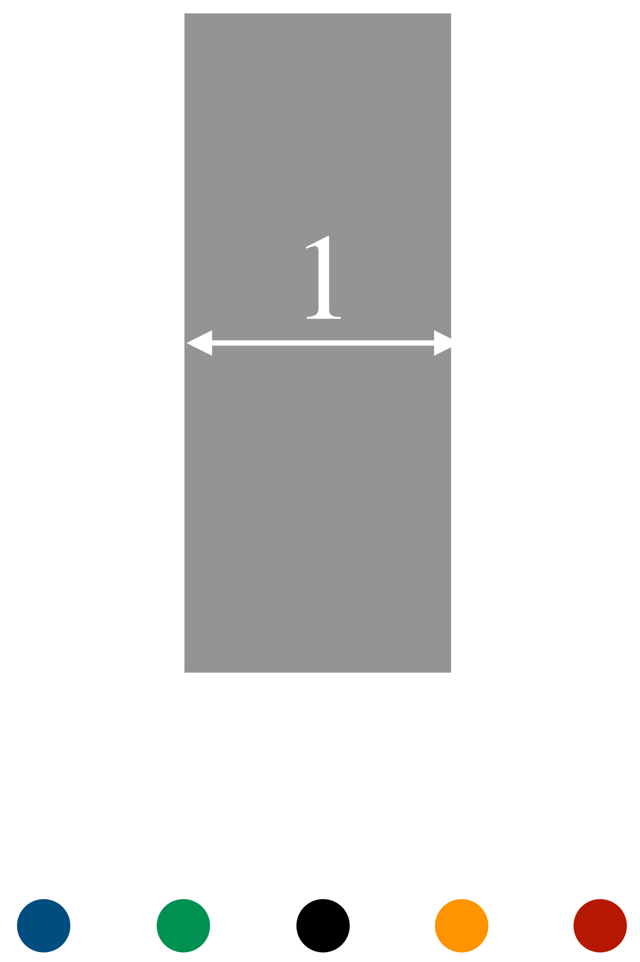

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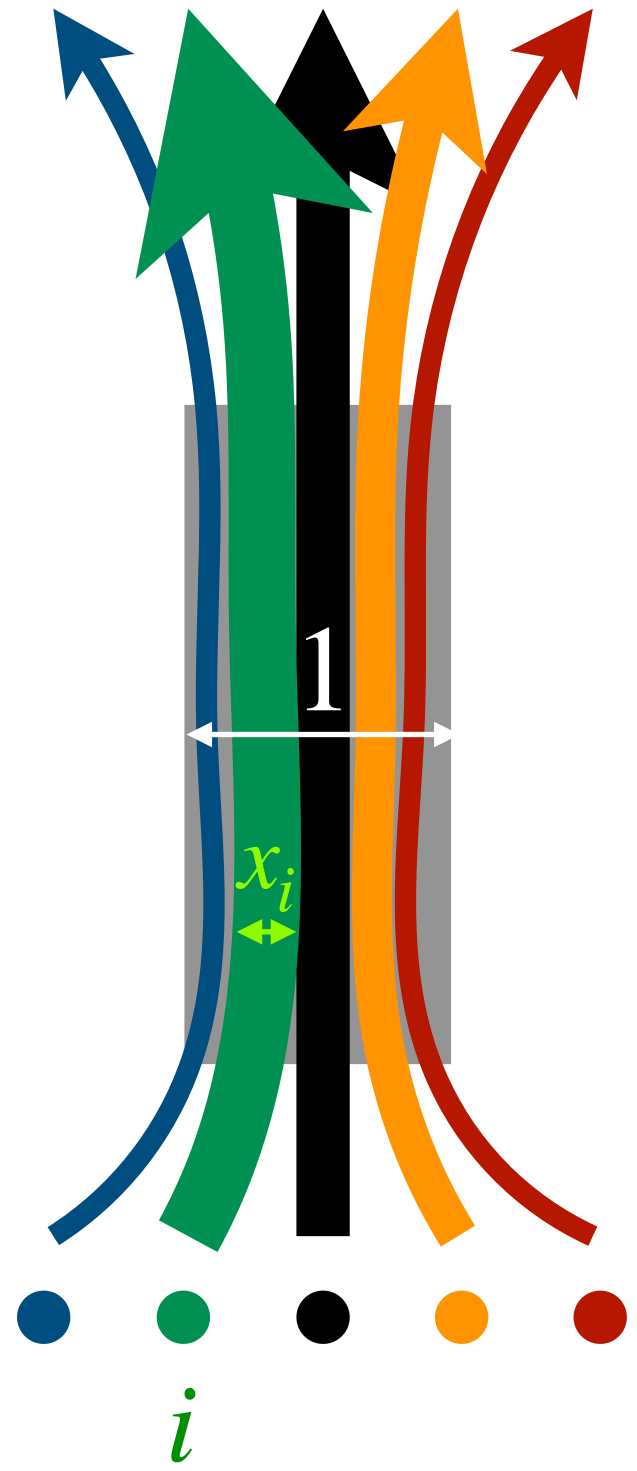

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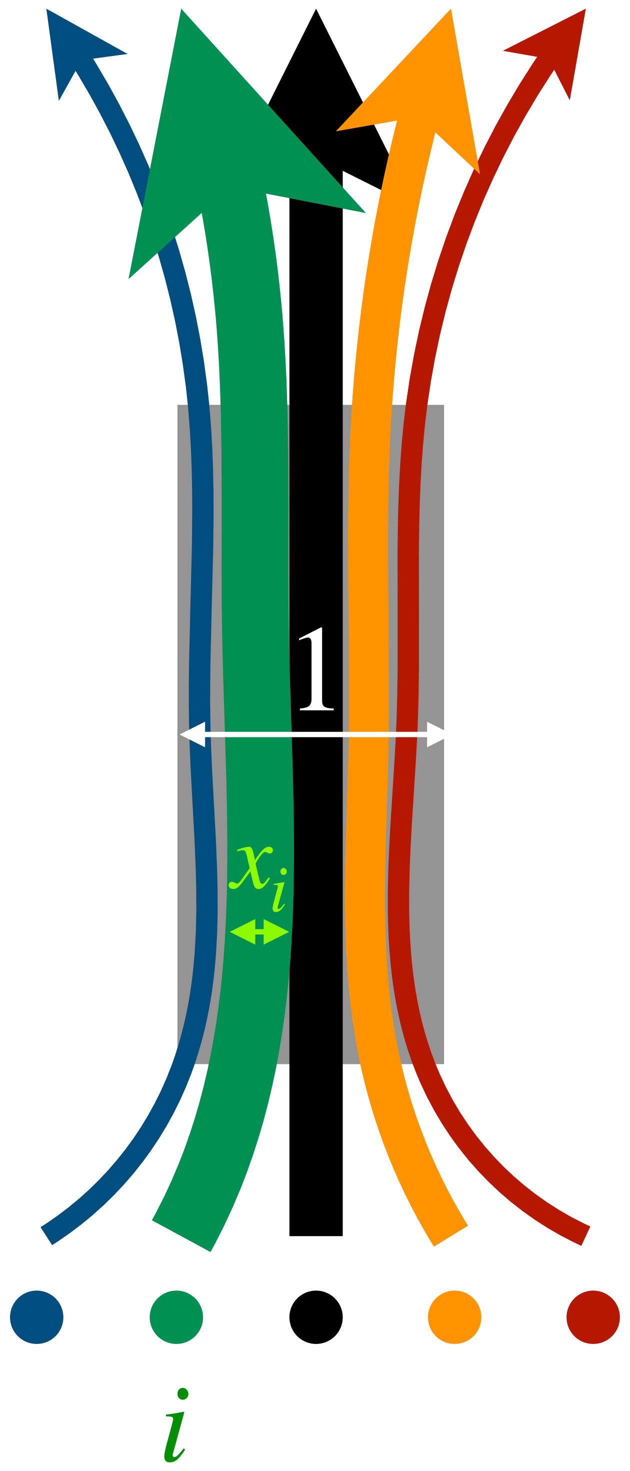

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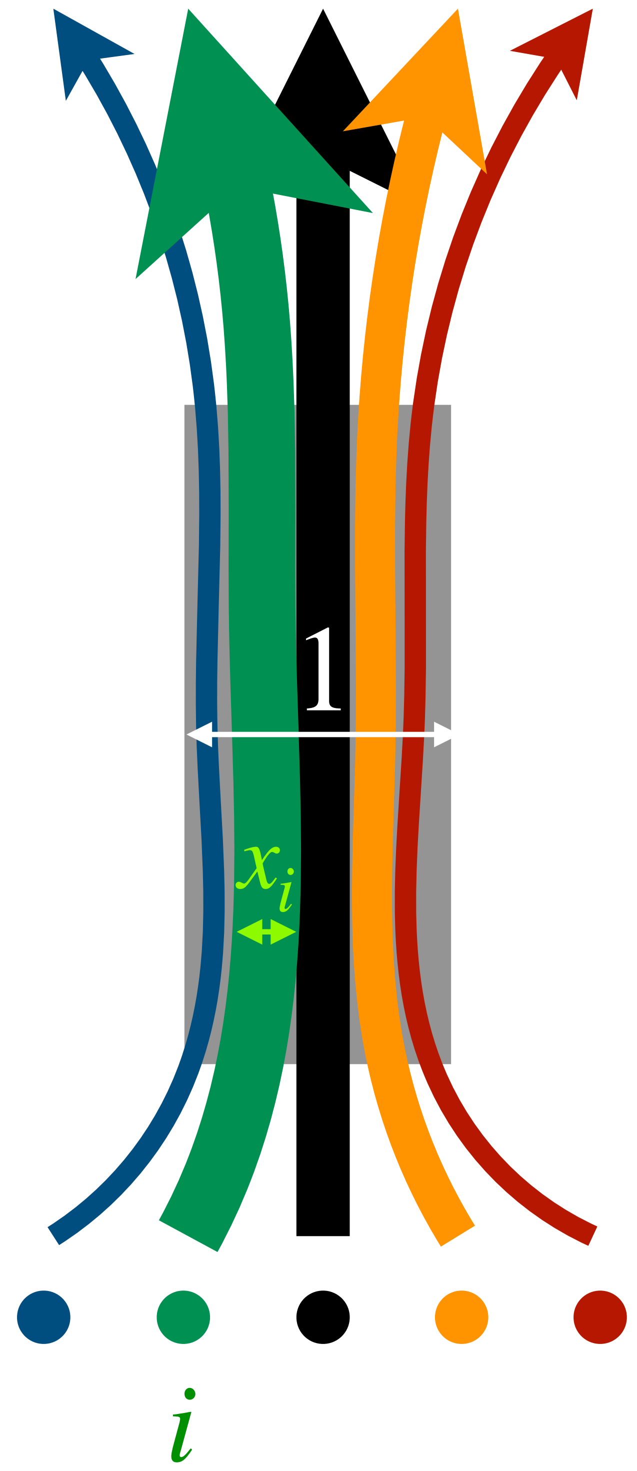

Tragedy of Commons — Stable solution

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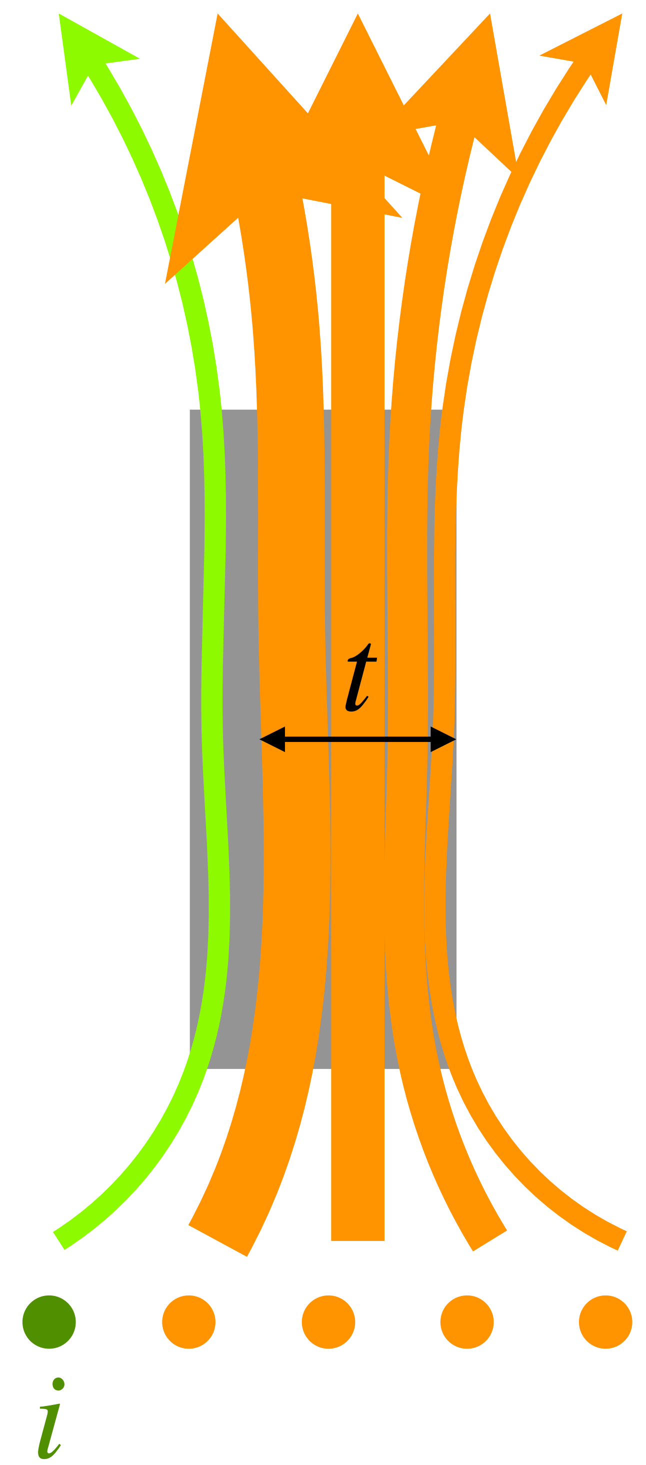
- Concentrate on player i. Let $t = \sum_{j \neq i} x_j < 1$ be the flow sent by all others



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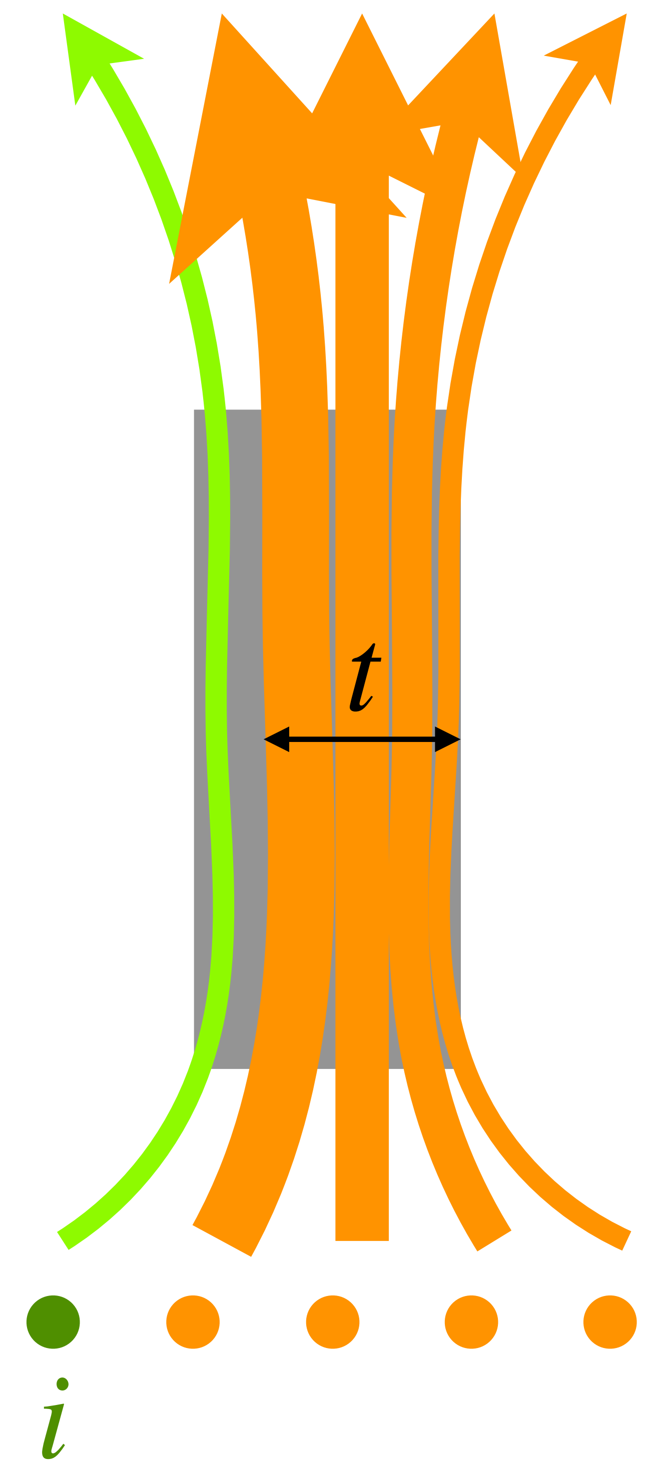

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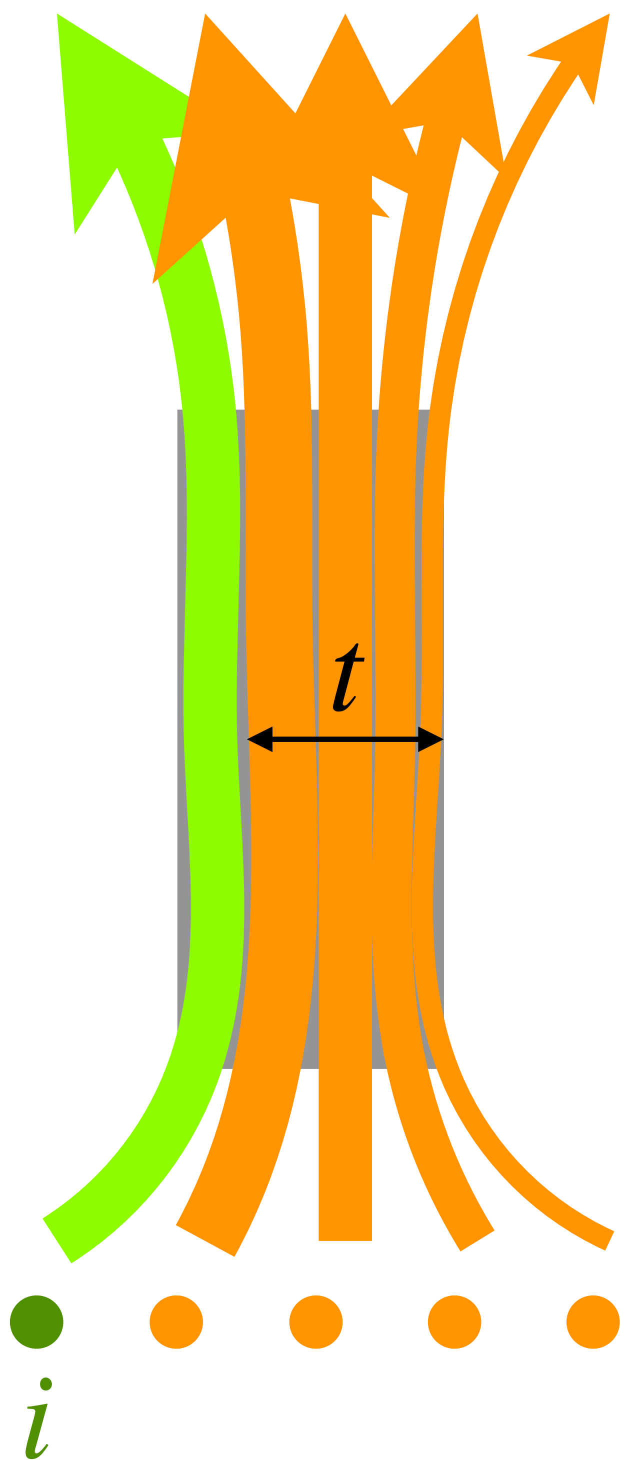

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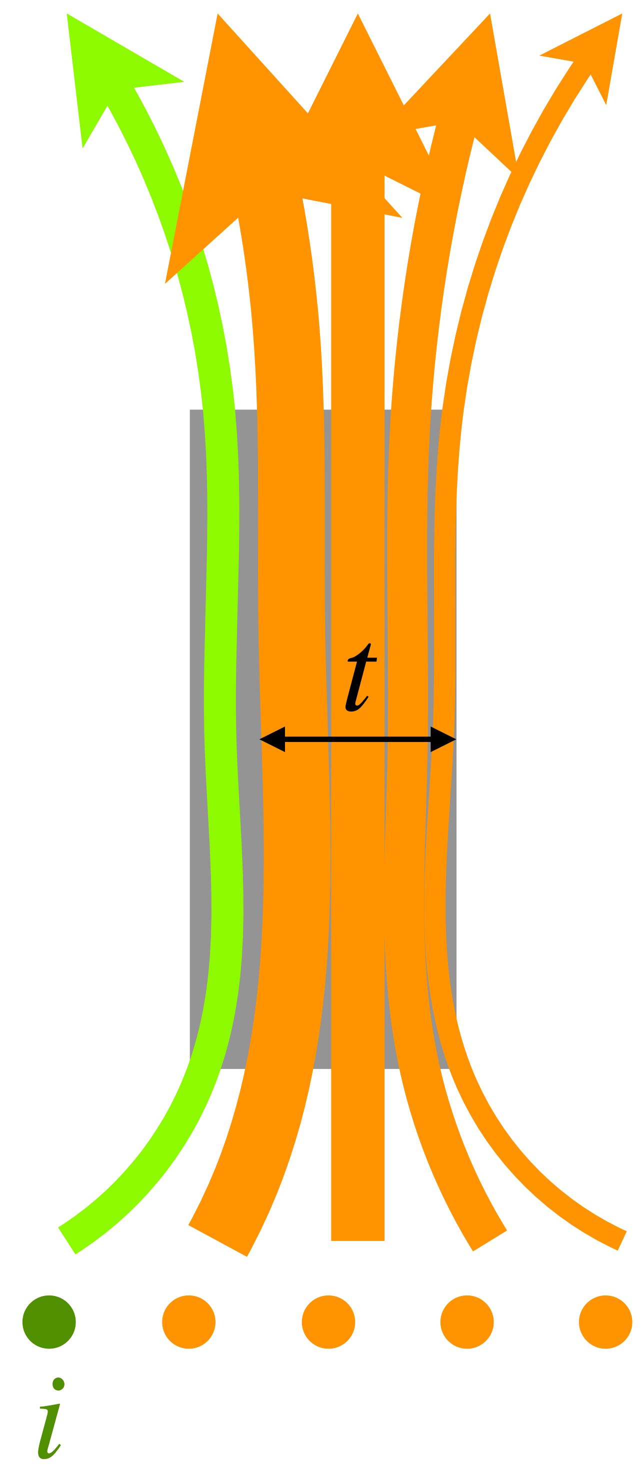

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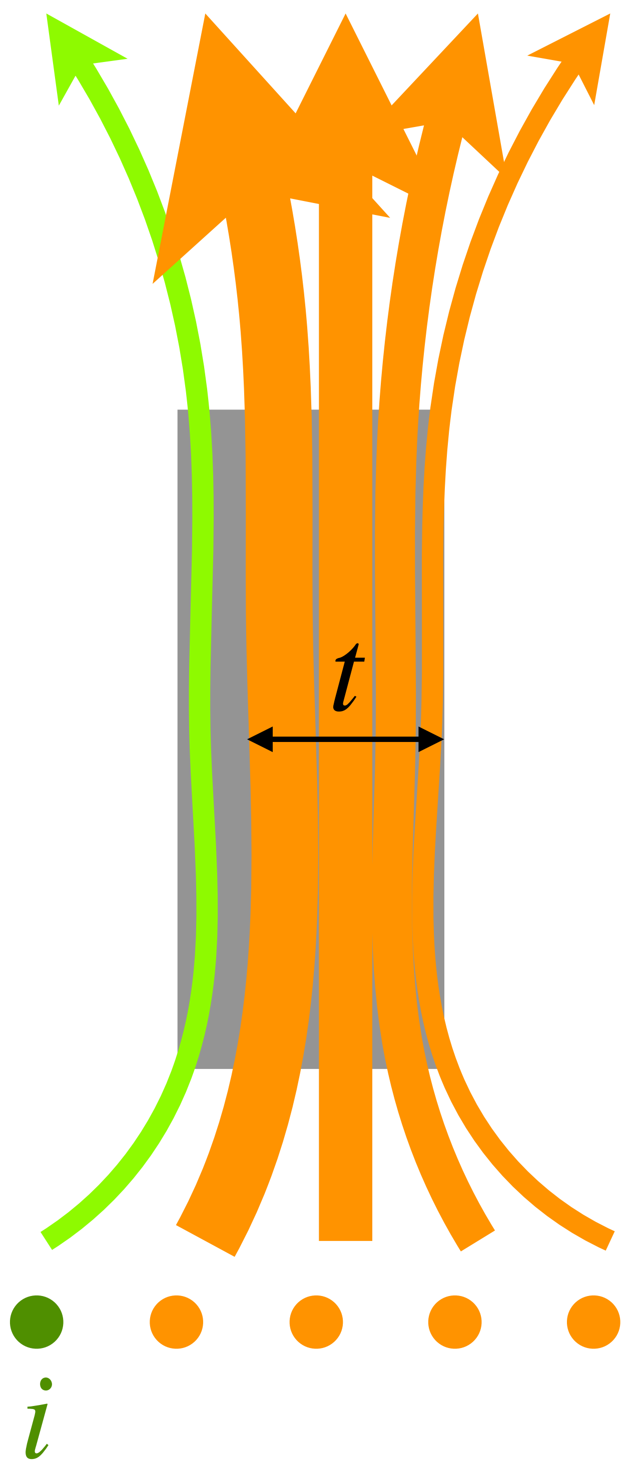

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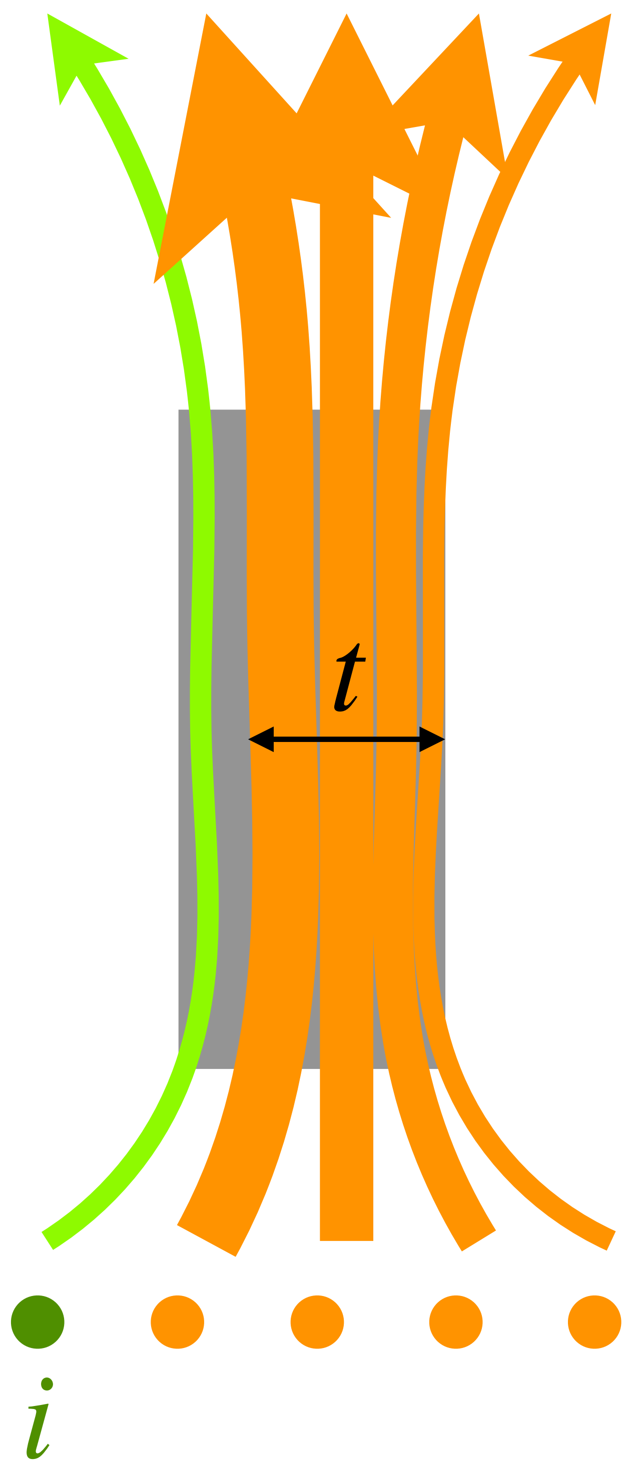
- A set of strategies is stable if all players are playing their optimal selfish strategy, given the strategies of all other players



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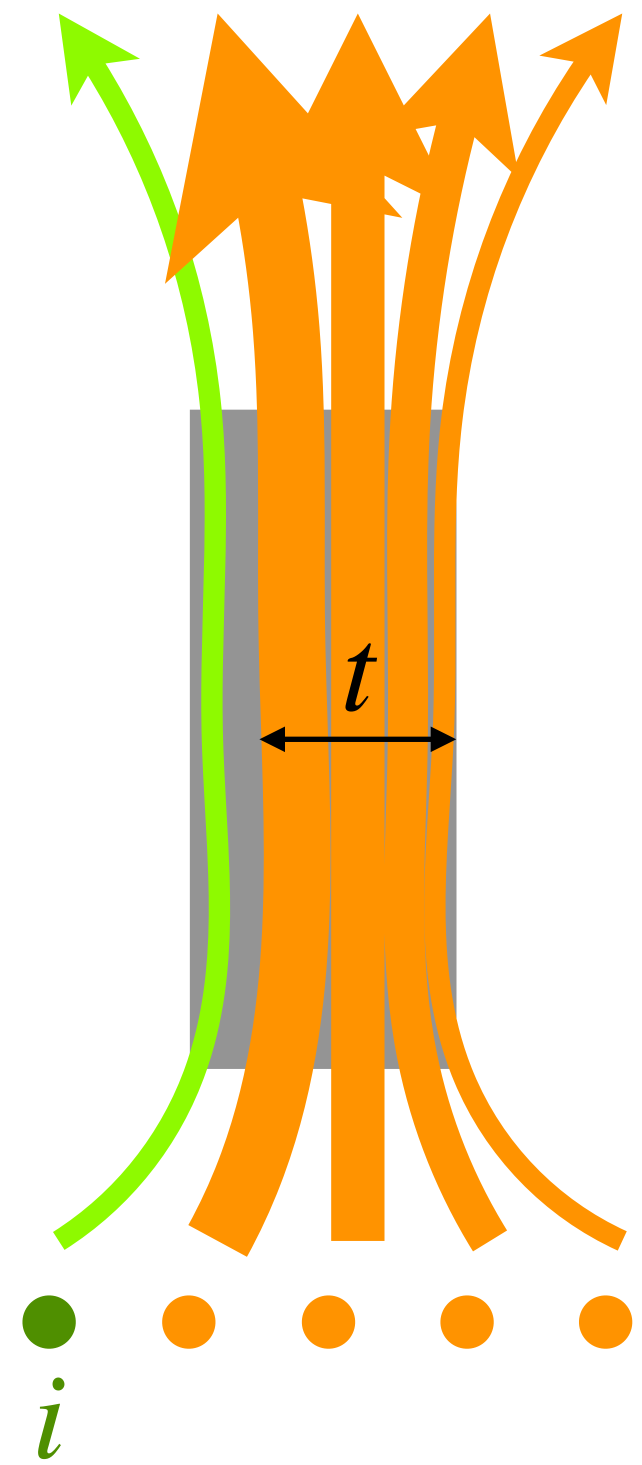

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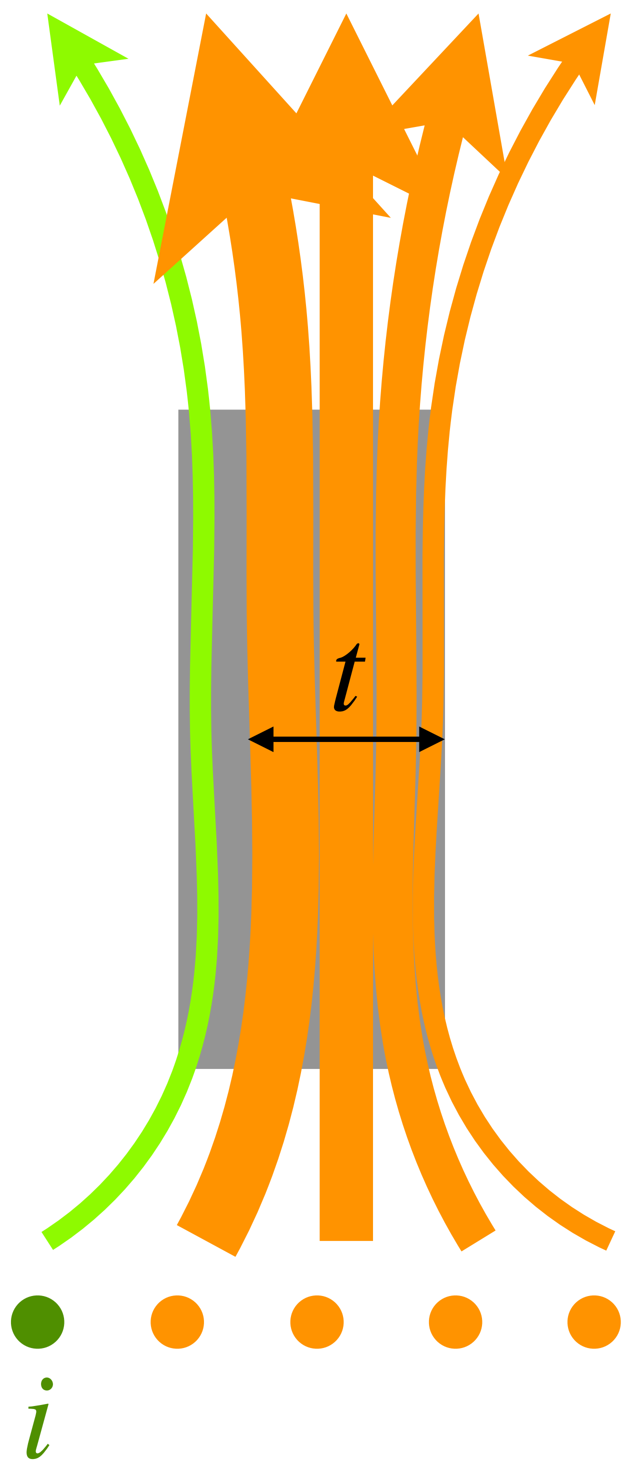
$$\Rightarrow x_i = \frac{1 - \Sigma_{j \neq i} x_j}{2} \text{ for all } i \quad \Rightarrow \Sigma_i x_i = \frac{n}{2} - \frac{n-1}{2} \cdot \Sigma_i x_i$$



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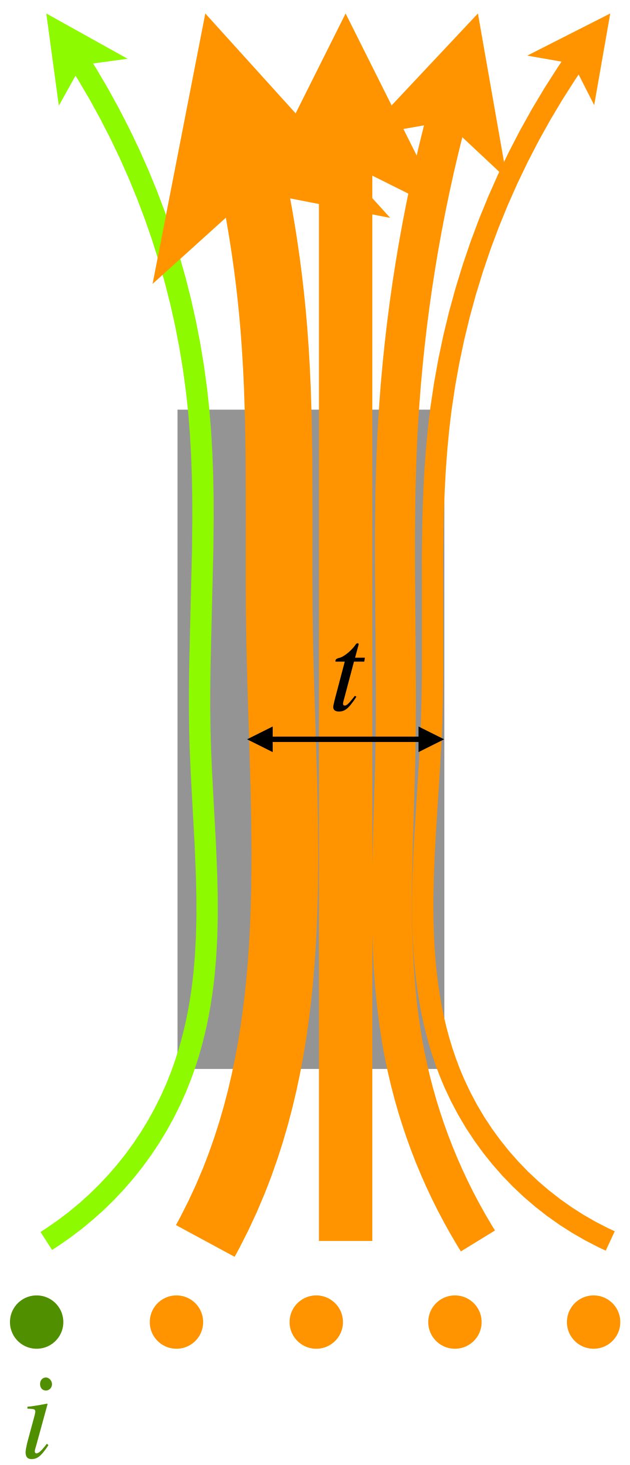
$$\Rightarrow x_i = \frac{1}{n+1} \text{ for all } i$$



Tragedy of Commons — Better solution

- Selfish strategy: $x_i = \dfrac{1}{n+1}$ for all i
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- (Centralized) better strategy: if the total bandwidth used is $\frac{1}{2} \cdot \frac{n}{n+1}$:
  - $x_i = \frac{1}{2(n+1)}$ for each player i, and the value of each player is $\frac{1}{2(n+1)} \cdot (1 - \frac{n}{2(n+1)}) = \frac{n+2}{4(n+1)^2}$
  - The new value is $\frac{n+2}{4}$ times the old value (!!)

What happened

- Self-interested behavior in a decentralized environment can decrease the overall performance:
  - Agents are selfish (Prisoner's dilemma)
  - Agents cannot communicate (Evening together, tragedy of commons)

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Games: Formal Definitions

- A game consists of a set of n self-interested *players*, $\{1,2,\cdots,n\}$
- Each player i selects a *strategy* s_i
- The *vector of strategies* $\vec{s} = (s_1, s_2, \cdots, s_n)$ selected by the players determine the outcome for each player
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c_A(confess, silent) $= 1$

c_B(confess, silent) $= 5$

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u_B(softball, softball) $= 5$

u_G(softball, softball) $= 6$

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  - The channel maximum capacity is 1, but the quality of the channel deteriorates with the total bandwidth used
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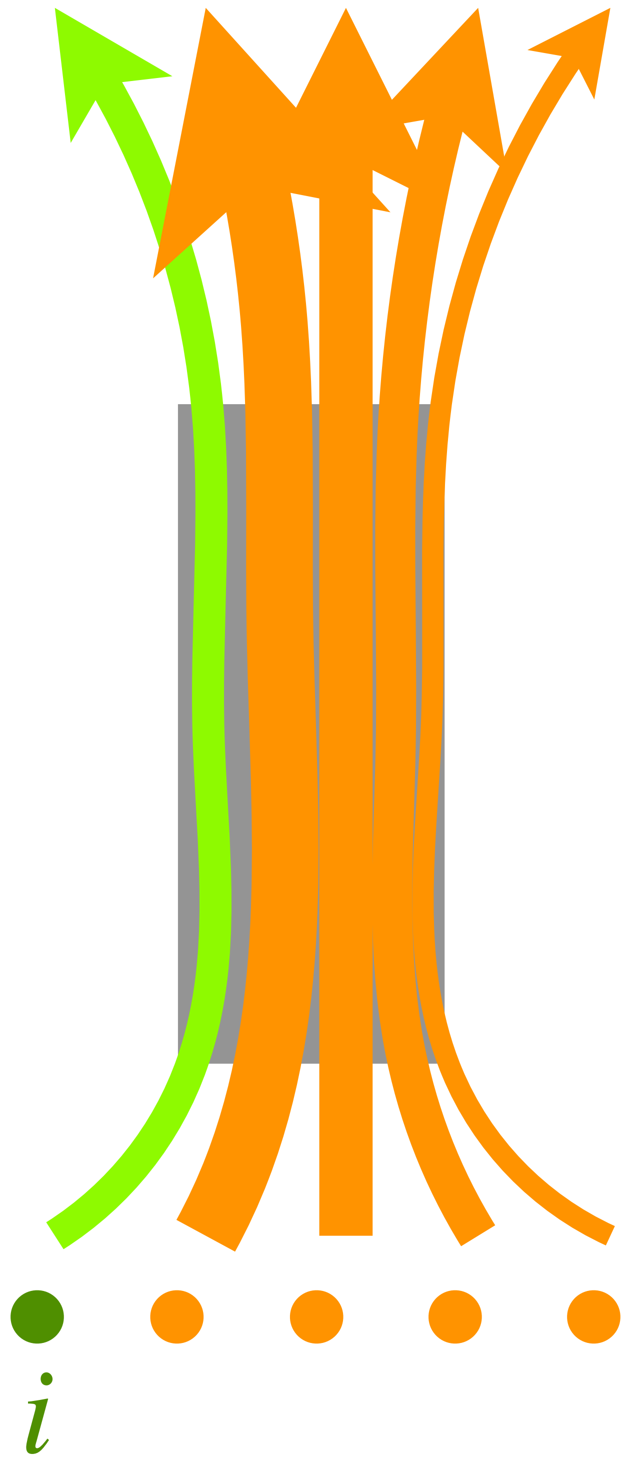

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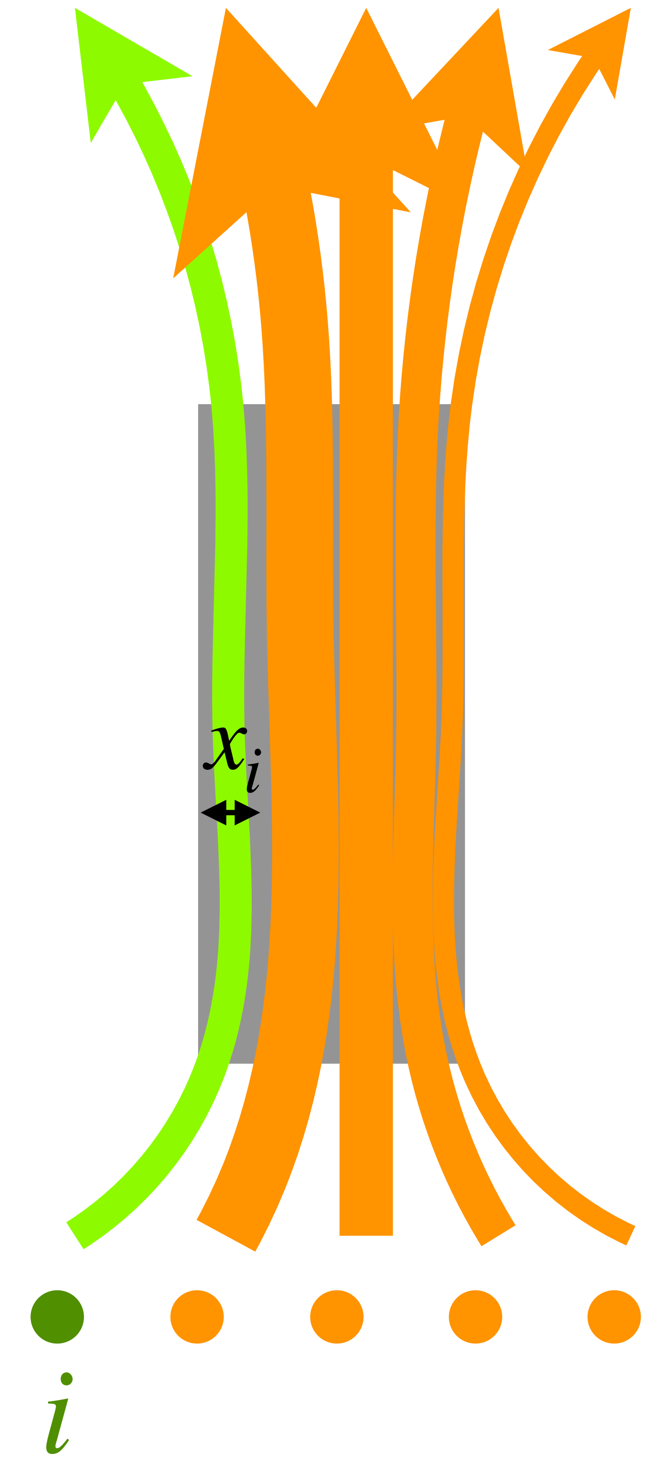

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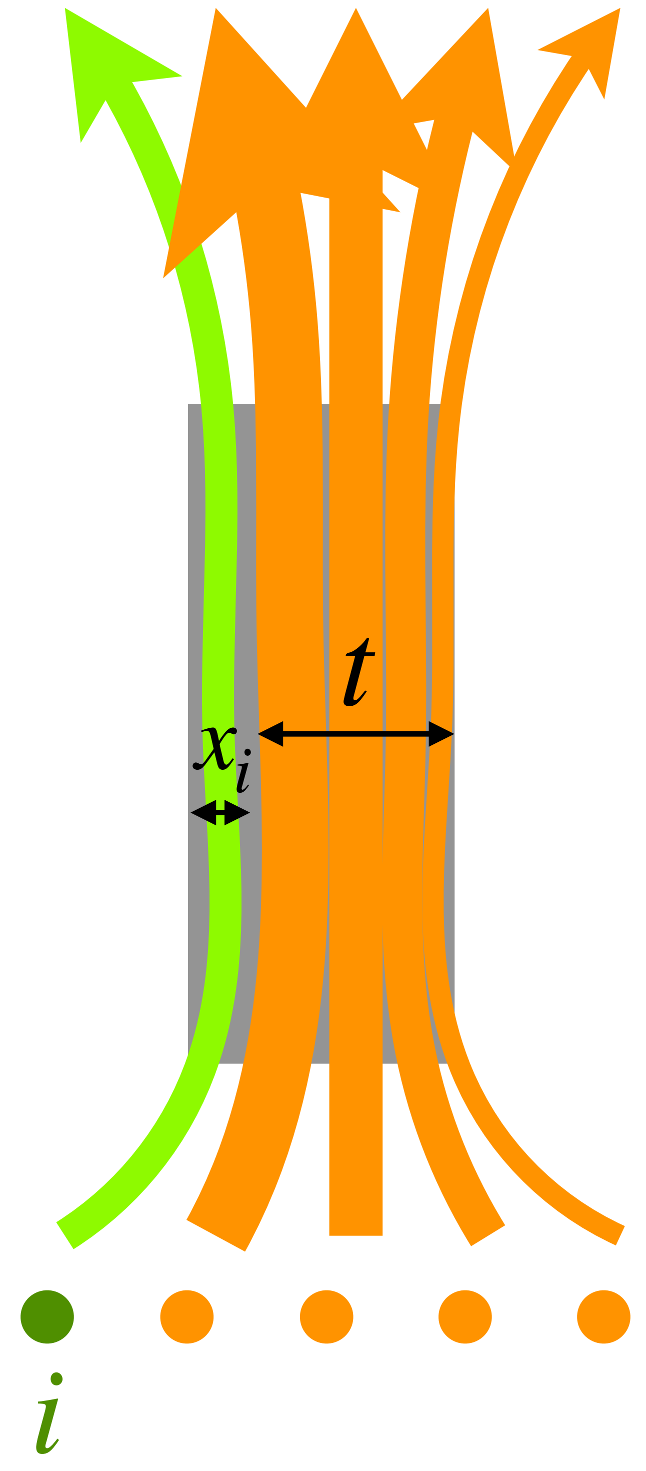

strategy: $x_i \in [0,1]$

$u_i(x_i, t) = x_i(1 - t - x_i)$

Outline

- Fundamental concepts
    - Game, players, strategies, payoffs/costs
- Nash Equilibrium
- Price of Anarchy
    - Selfish load balancing
- Mechanism design
    - Auction
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Nash Equilibrium

- Player i *(weakly) prefers* $\overrightarrow{s_x}$ to $\overrightarrow{s_y}$ if i prefers $\overrightarrow{s_x}$ to $\overrightarrow{s_y}$ or considers them as equally good outcomes. That is, $u_i(\overrightarrow{s_x}) \geq u_i(\overrightarrow{s_y})$
  - $\overrightarrow{s_{-i}} = (s_1, s_2, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n)$
  - $\vec{s} = (s_1, s_2, \cdots, s_{i-1}, s_i, s_{i+1}, \cdots, s_n) = (s_i, \overrightarrow{s_{-i}})$

$\vec{s} =$ (confess, silent)

$\overrightarrow{s_{-A}} =$ (silent)

A B	confess	silent
confess	4 4	5 1
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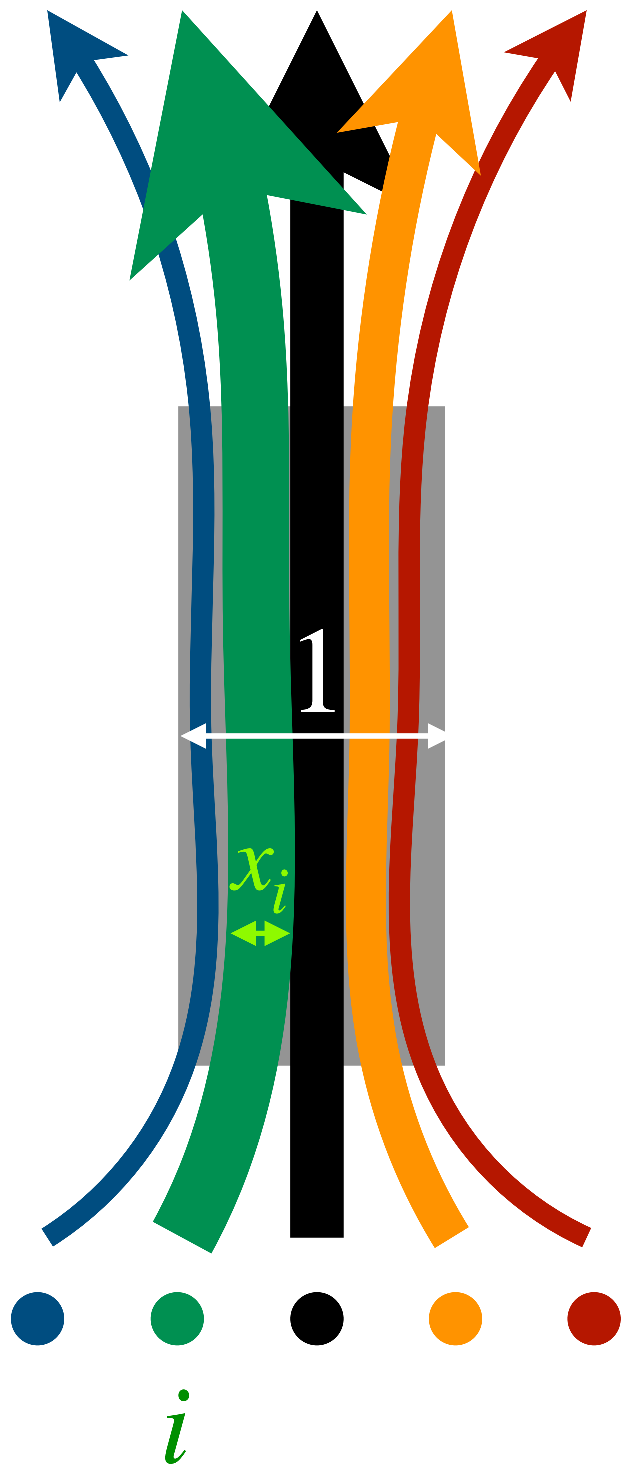
$$\vec{s} = (0.08, 0.25, 0.2, 0.15, 0.08)$$
$$\overrightarrow{s_{-2}} = (0.08, 0.2, 0.15, 0.08)$$



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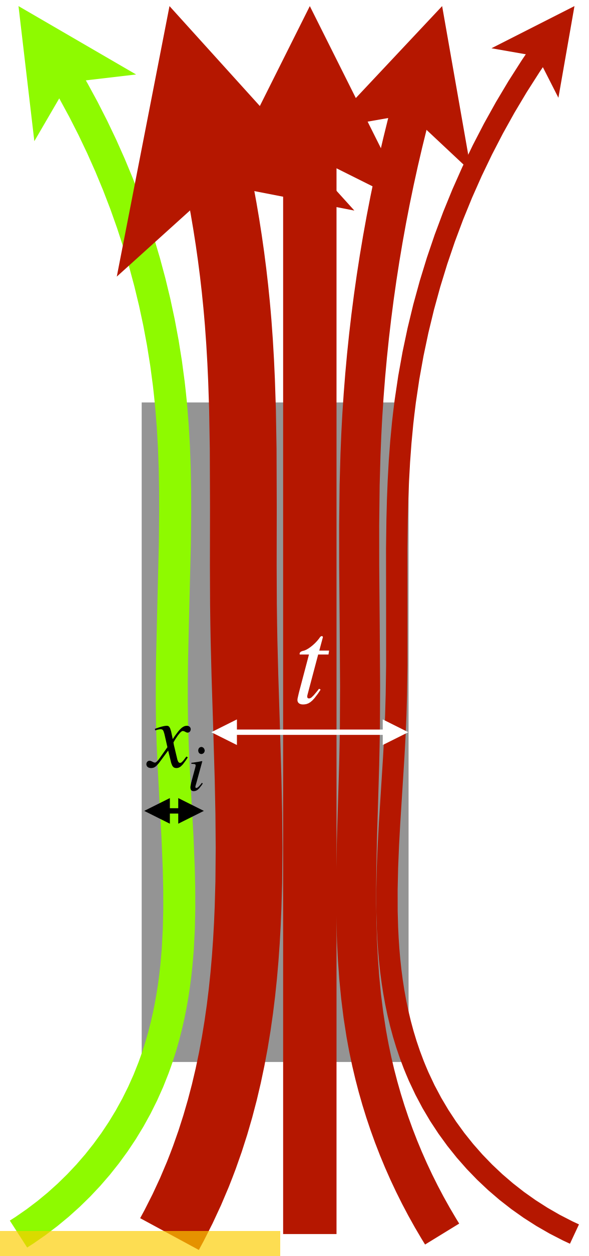
- A strategy vector $\vec{s}$ is a **Nash equilibrium** if

for all players i and each alternate strategy s_i':

$$u_i(s_i, \overrightarrow{s_{-i}}) \geq u_i(s_i', \overrightarrow{s_{-i}})$$

$$(\text{or, } c_i(s_i, \overrightarrow{s_{-i}}) \leq c_i(s_i', \overrightarrow{s_{-i}}))$$

Nash Equilibrium



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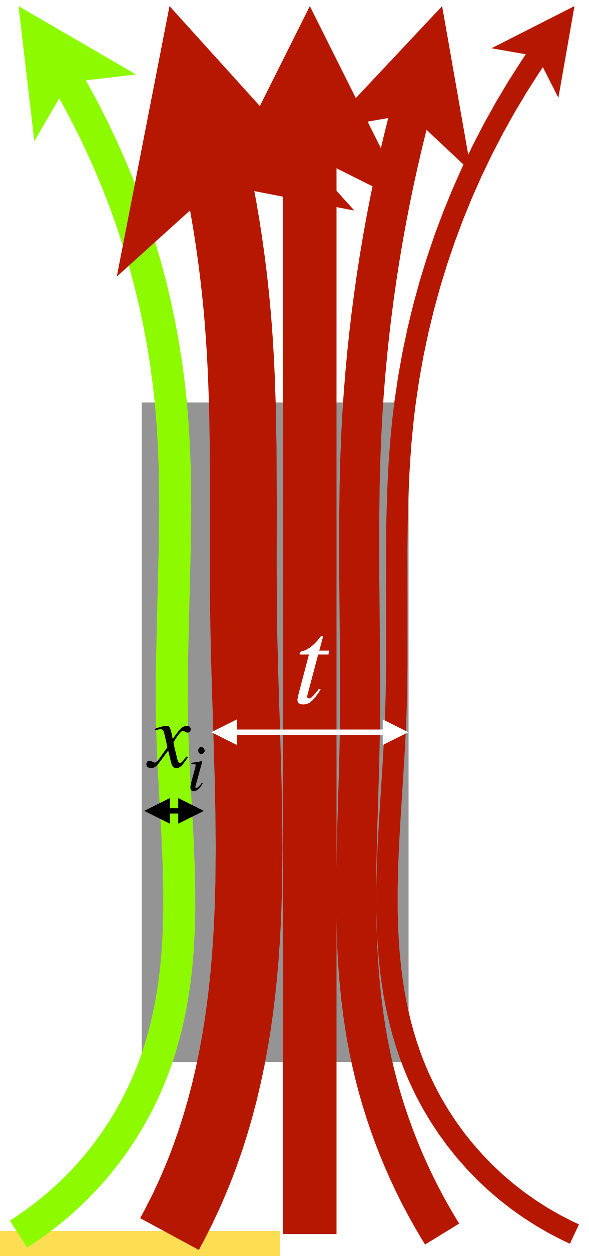

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only change strategy from s_i to s_i' when s_i' is strictly better

What happened

- Nash equilibrium: The stable state that no player can improve its wellbeing by changing its own strategy (given others’ strategies don’t change)

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Social Welfare/Cost

- Social choice: an aggregation of the preference of the different participants toward a single joint decision
- Let $\vec{s}$ be a preferences/strategies of the players
    - The social choice $f(\vec{s})$ is the action given $\vec{s}$, and it has a social welfare (or social cost)

Price of Anarchy (PoA)

- Measure the inefficiency of equilibria
- Given a game, let S_{NE} be its set of equilibria (all stable strategies), the **Price of Anarchy** is

$$\frac{\max_{\vec{s}} \text{Social Welfare}(\vec{s})}{\min_{\vec{s} \in S_{\text{NE}}} \text{Social Welfare}(\vec{s})}$$

or

$$\frac{\max_{\vec{s} \in S_{\text{NE}}} \text{Social Cost}(\vec{s})}{\min_{\vec{s}} \text{Social Cost}(\vec{s})}$$

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$$\text{PoA} = \frac{4+4}{2+2}$$

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G \ B	baseball	softball
baseball	6 5	1 1
softball	2 2	5 6

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B G	baseball	softball
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$$\text{PoA} = \frac{6+5}{5+6}$$

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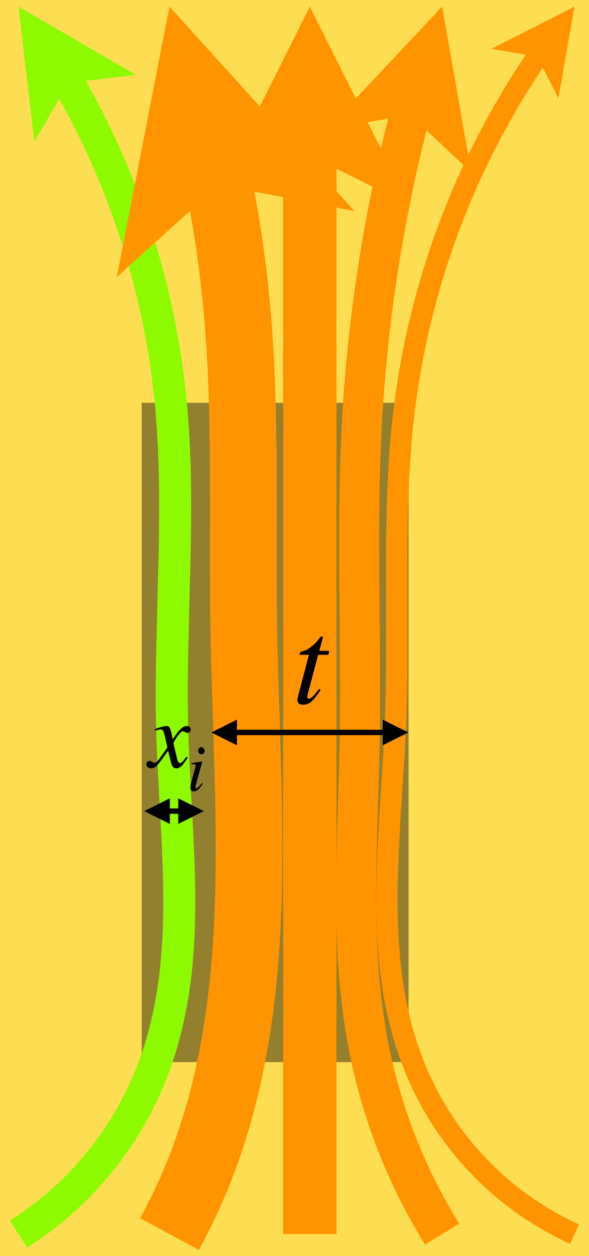

Tragedy of Commons — Better solution

- Selfish strategy: $x_i = \dfrac{1}{n+1}$ for all i
  - Total bandwidth used is $\dfrac{n}{n+1}$
  - For each player, the payoff is $x_i(1-\Sigma_j x_j) = \dfrac{1}{n+1}(1-\dfrac{n}{n+1}) = \dfrac{1}{(n+1)^2}$
- (Centralized) better strategy: if the total bandwidth used is $\dfrac{1}{2}\cdot\dfrac{n}{n+1}$:
  - $x_i = \dfrac{1}{2(n+1)}$ for each player i, and the payoff of each player is $\dfrac{1}{2(n+1)}\cdot(1-\dfrac{n}{2(n+1)}) = \dfrac{n+2}{4(n+1)^2}$
  - The new value is $\dfrac{n+2}{4}$ times the old value (!!)

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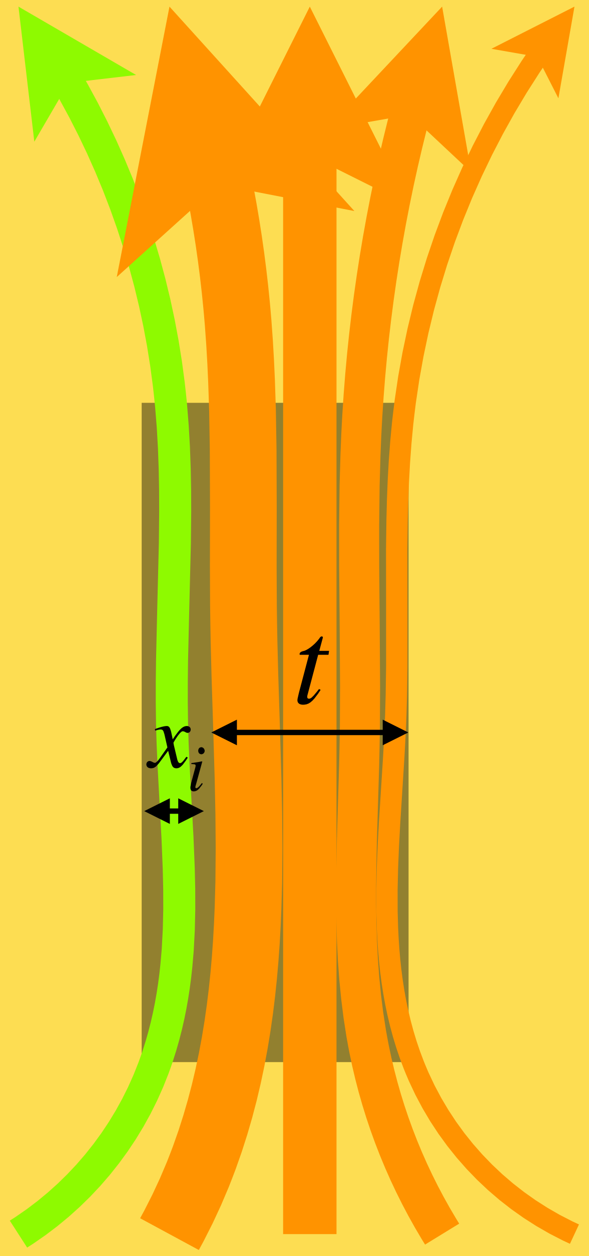

$$\text{PoA} \geq \frac{\frac{n(n+2)}{4(n+1)^2}}{\frac{n}{(n+1)^2}} = \frac{n+2}{4}$$

What happened

- Price of Anarchy measures the performance loss due to decentralization in the worst case

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Load Balancing Game

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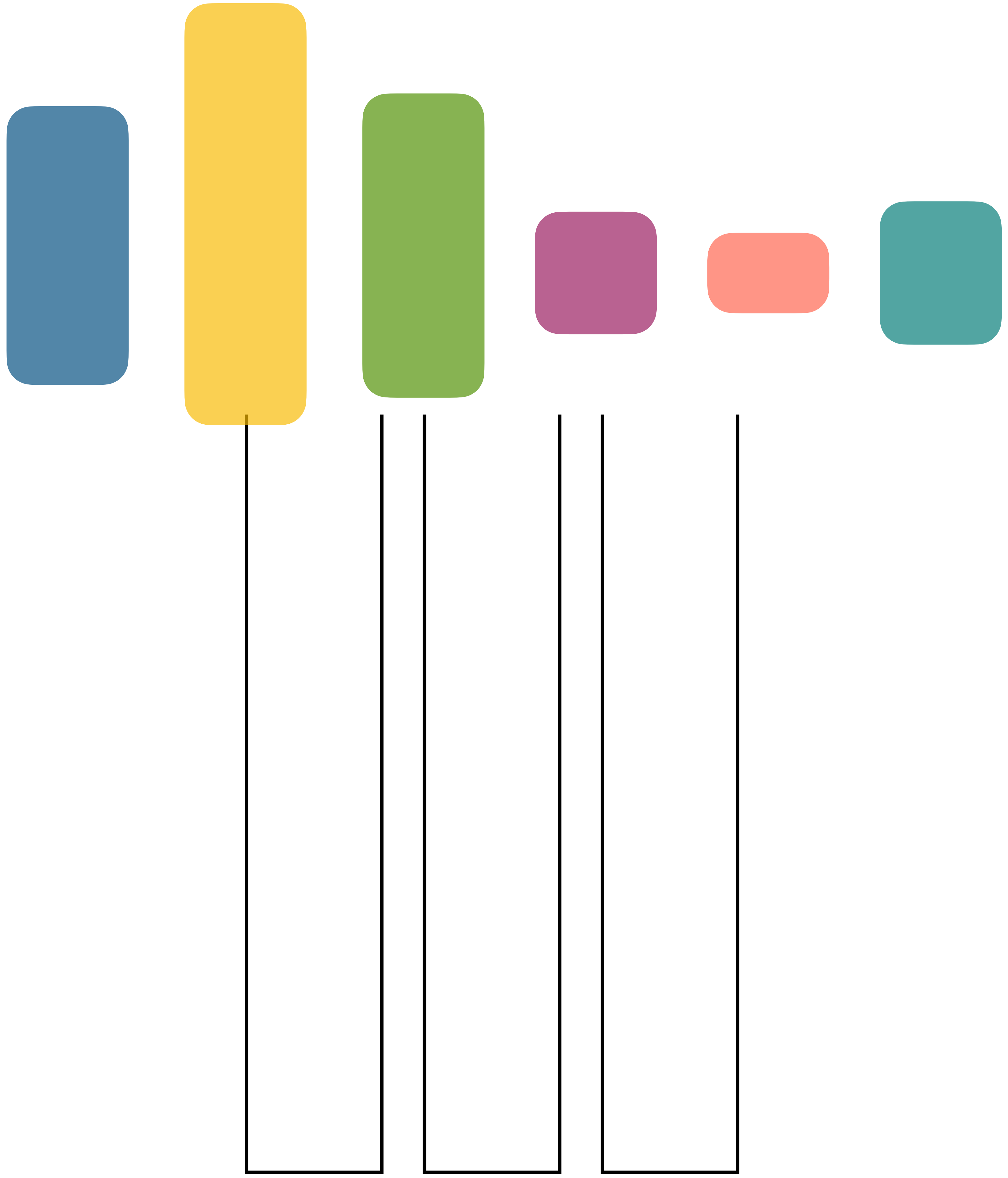

Load Balancing Game

- There are n jobs, each has processing time p_i and belongs to a self-interested player i
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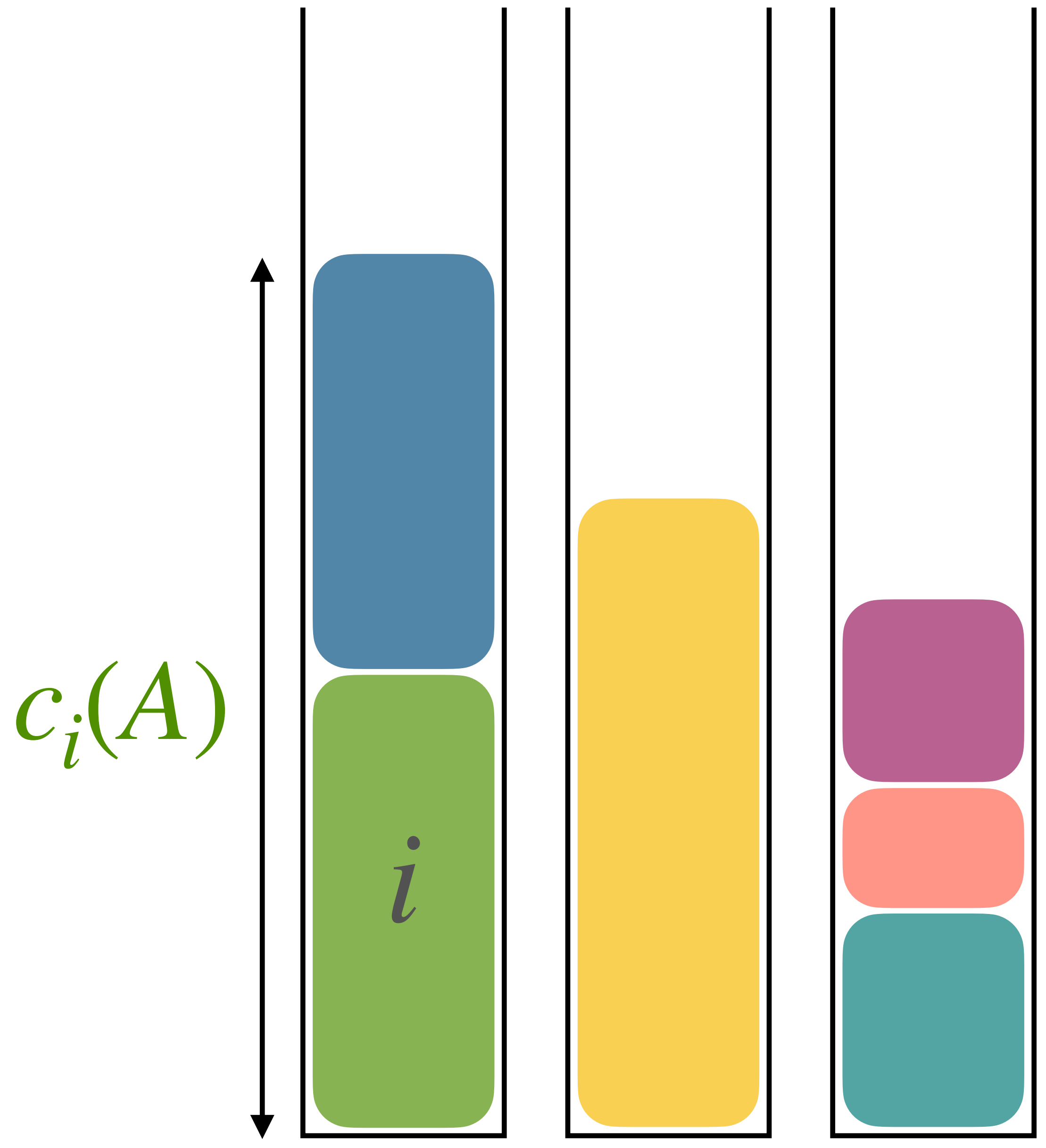
- Game: the players want to schedule their jobs on the lowest-loaded machine
    - Load of machine k: $\ell_k = \Sigma_{j \text{ is assigned to machine } k}\, p_j$



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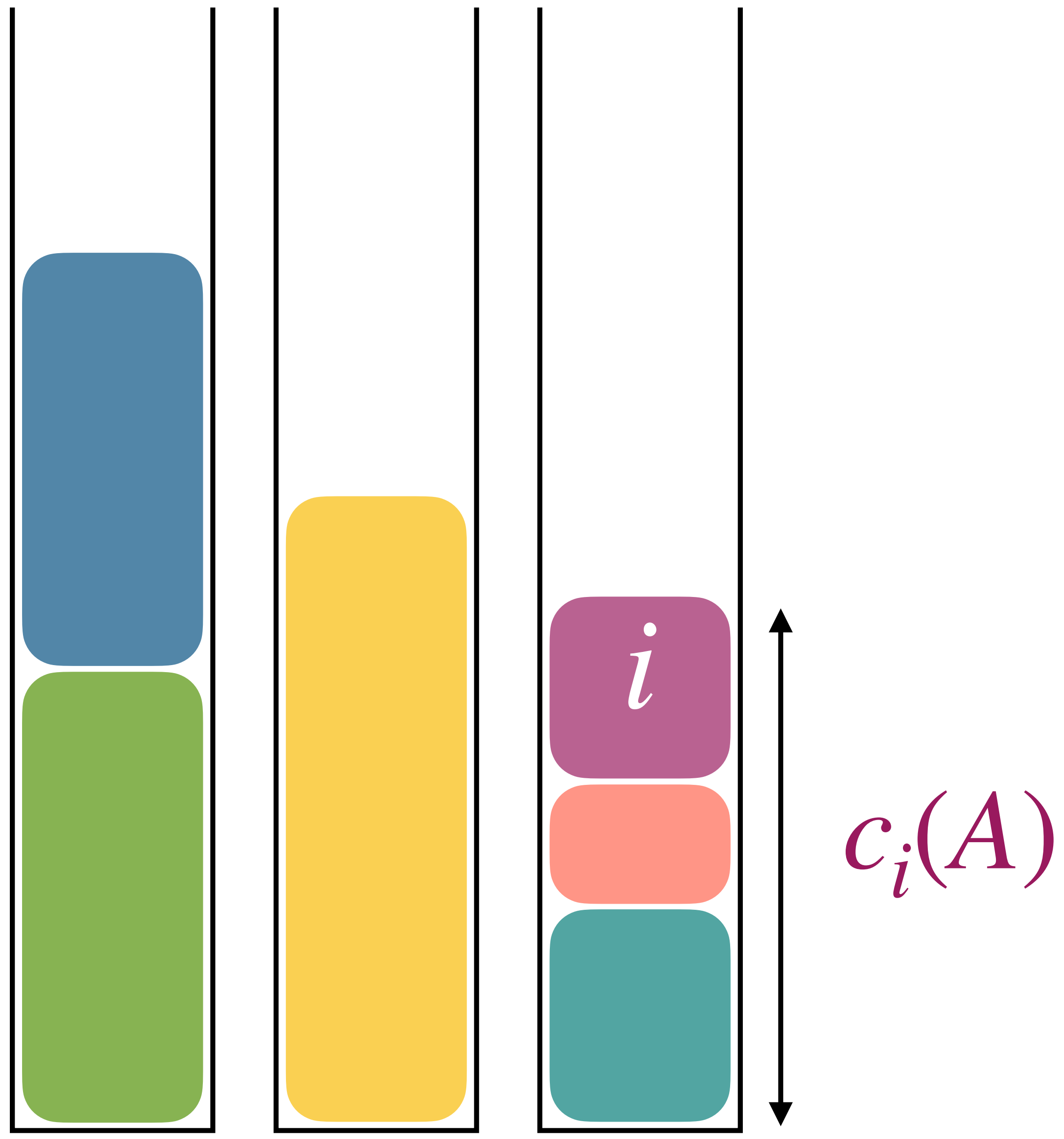

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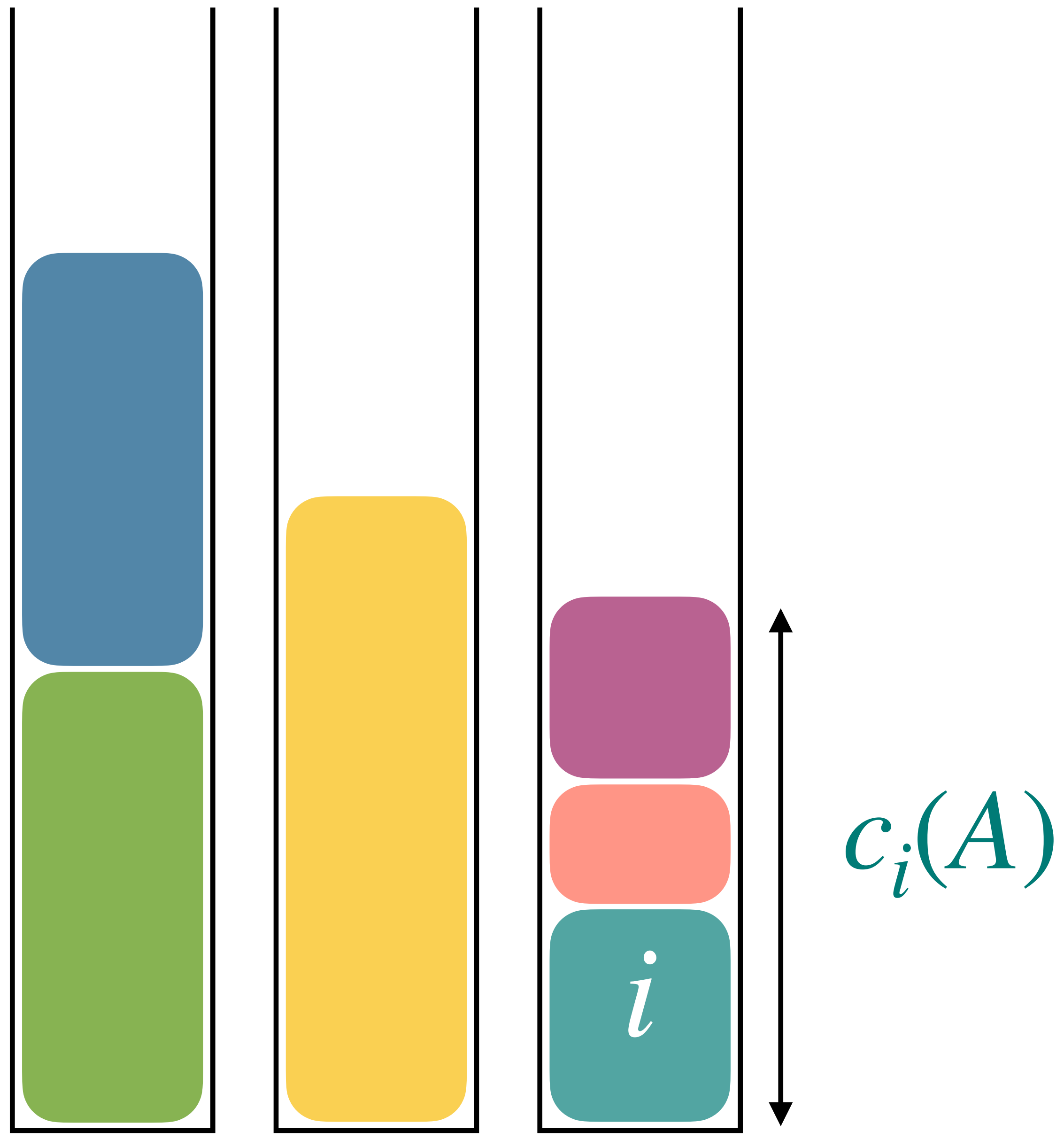

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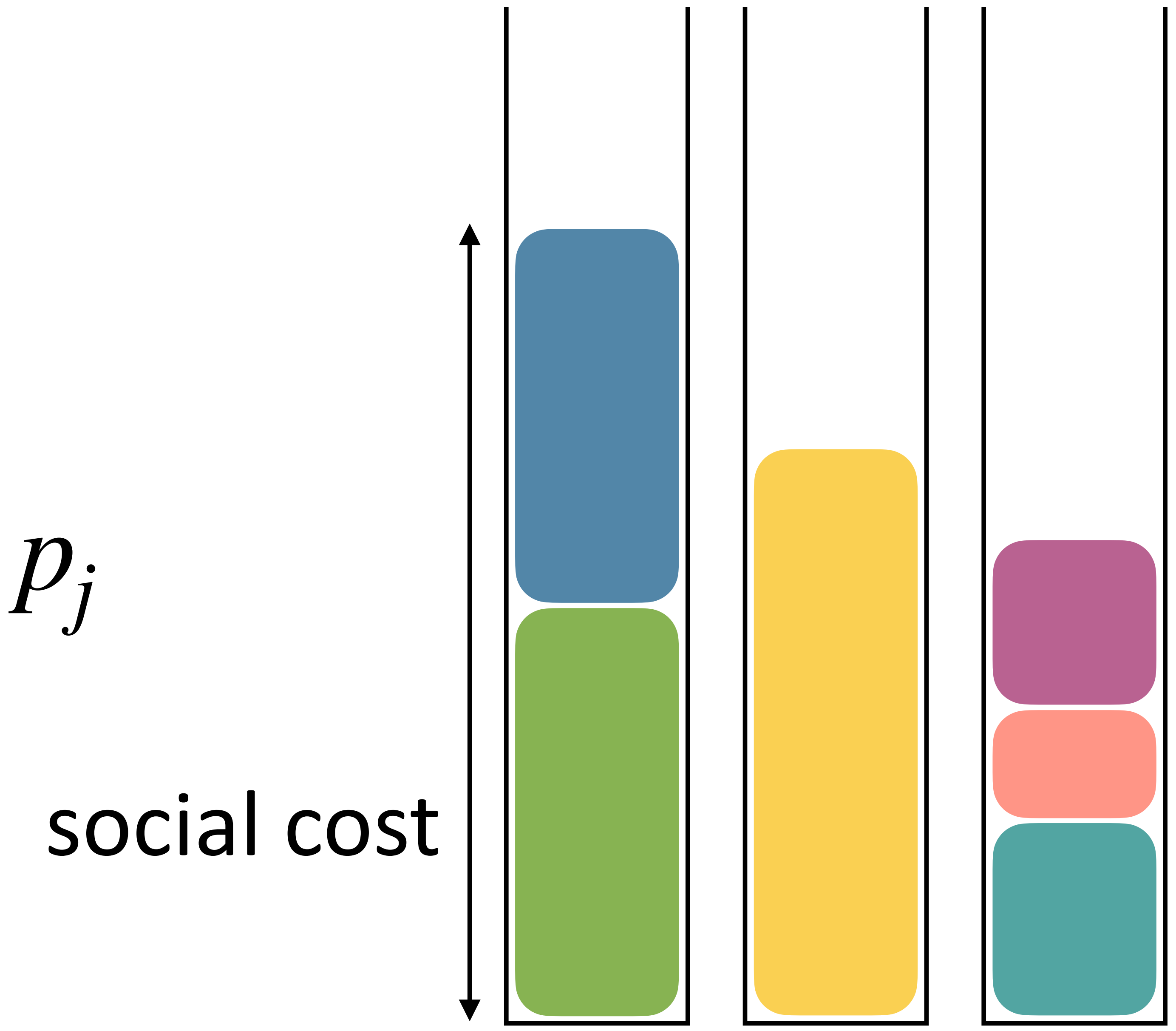

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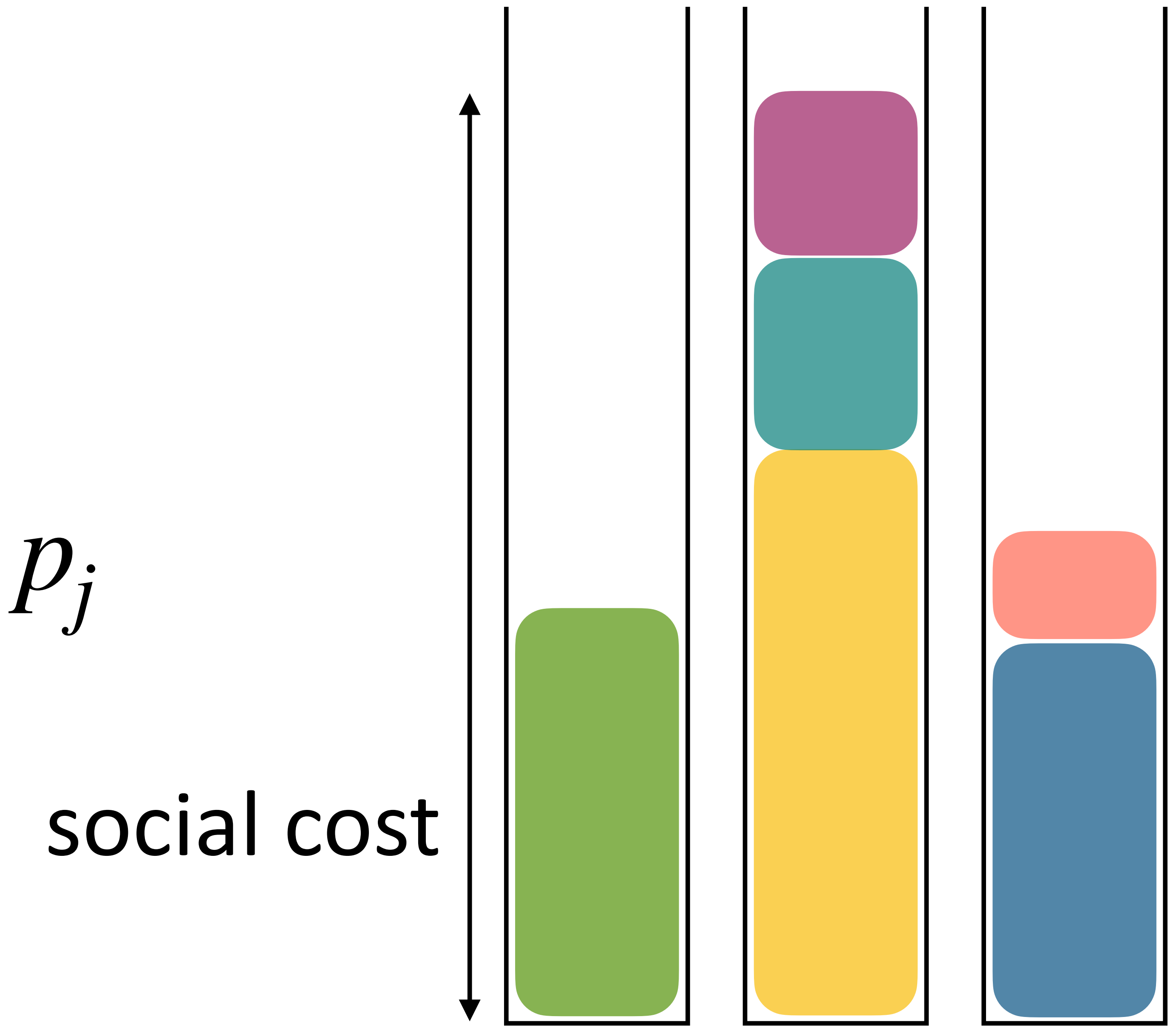

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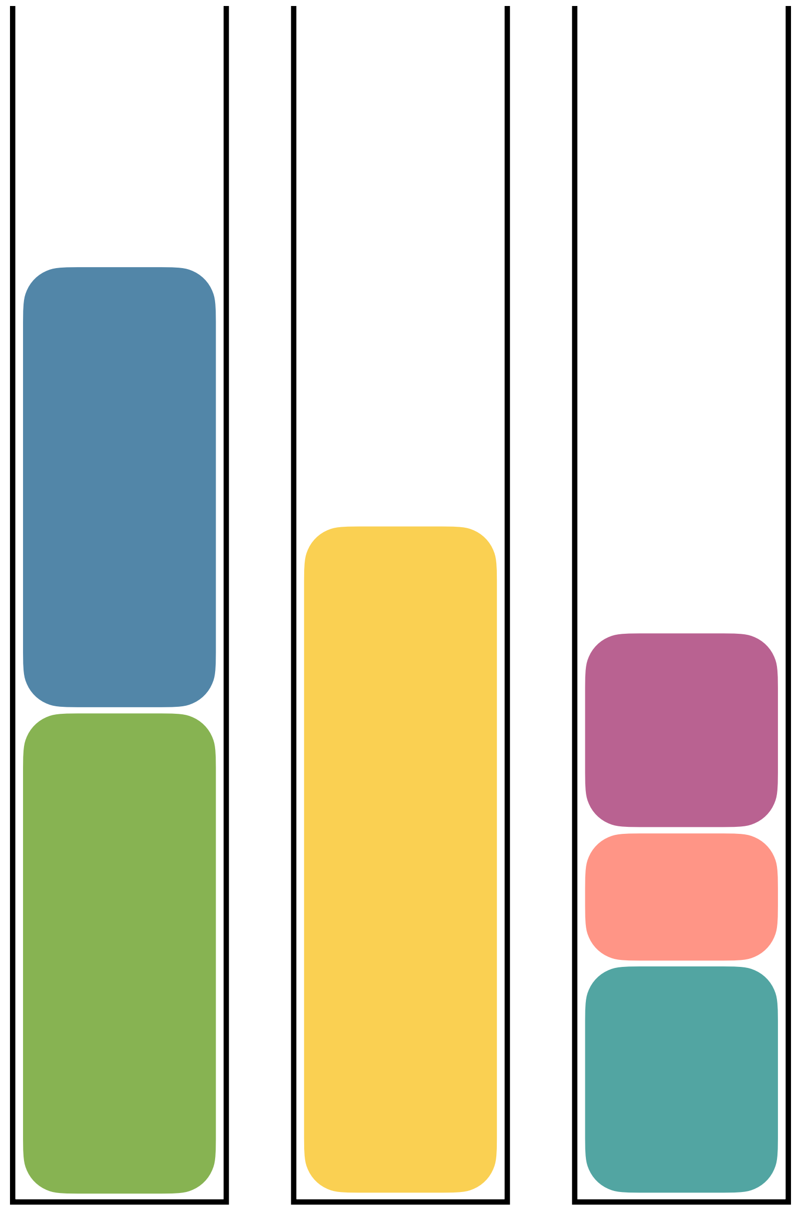

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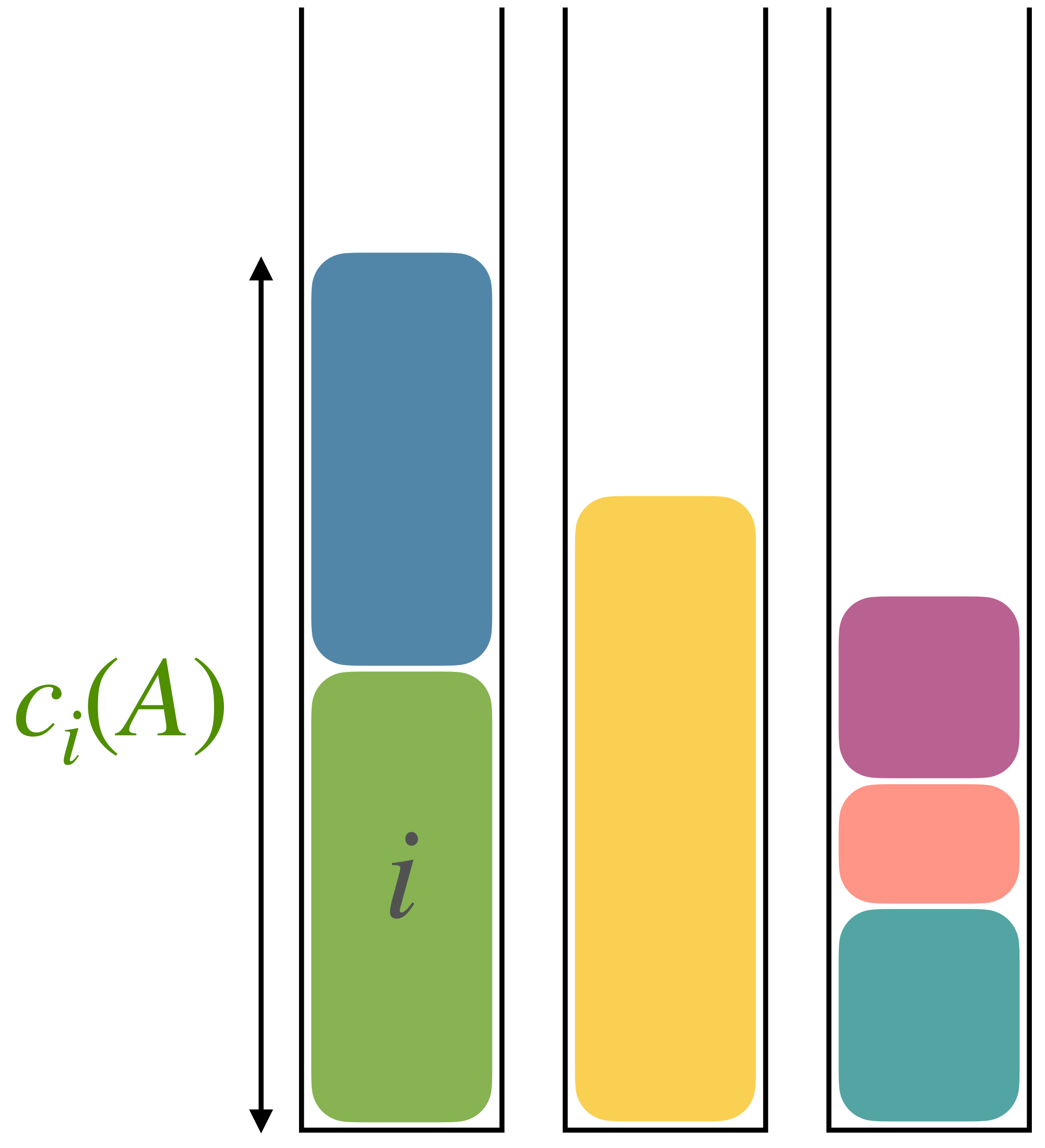

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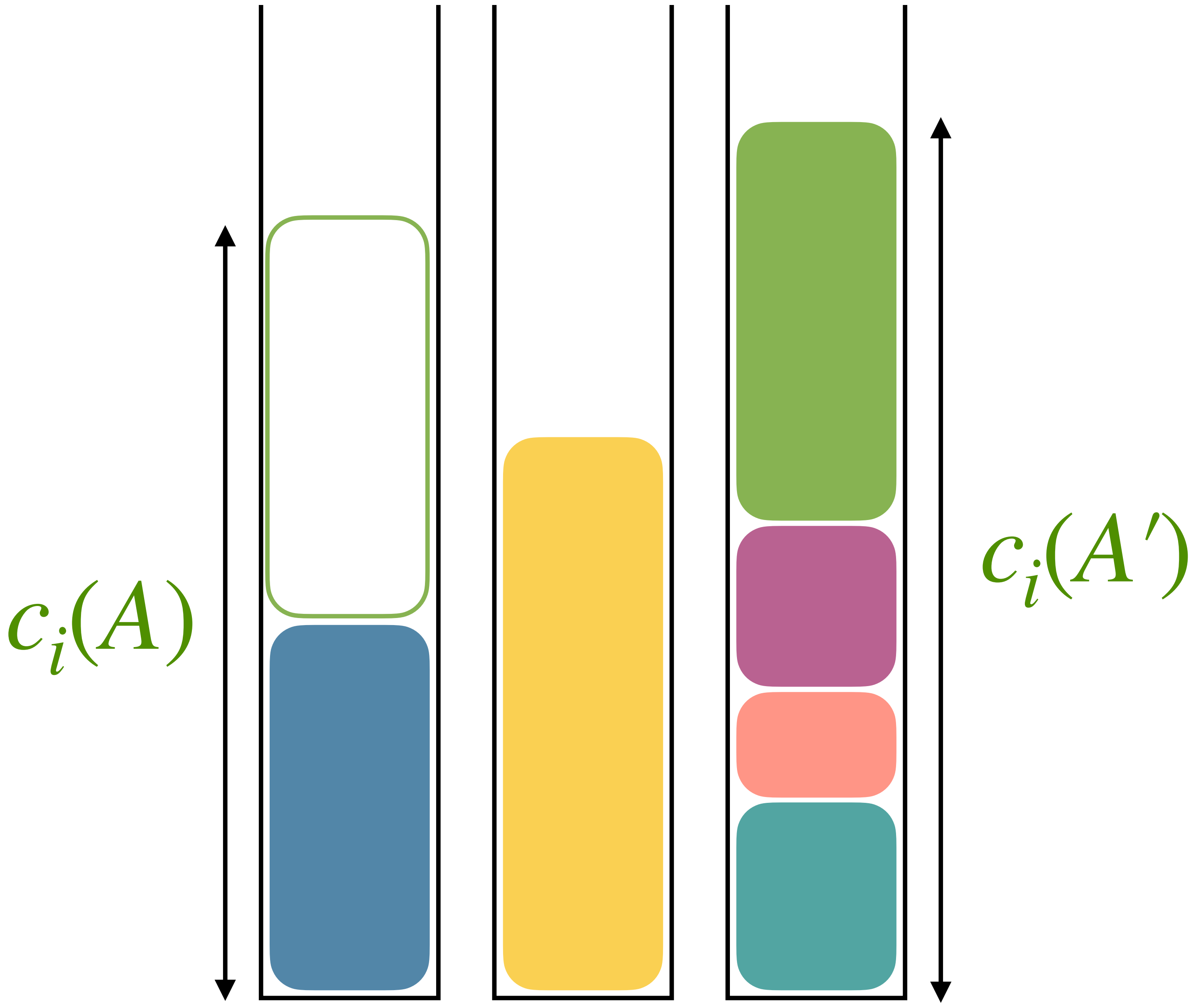

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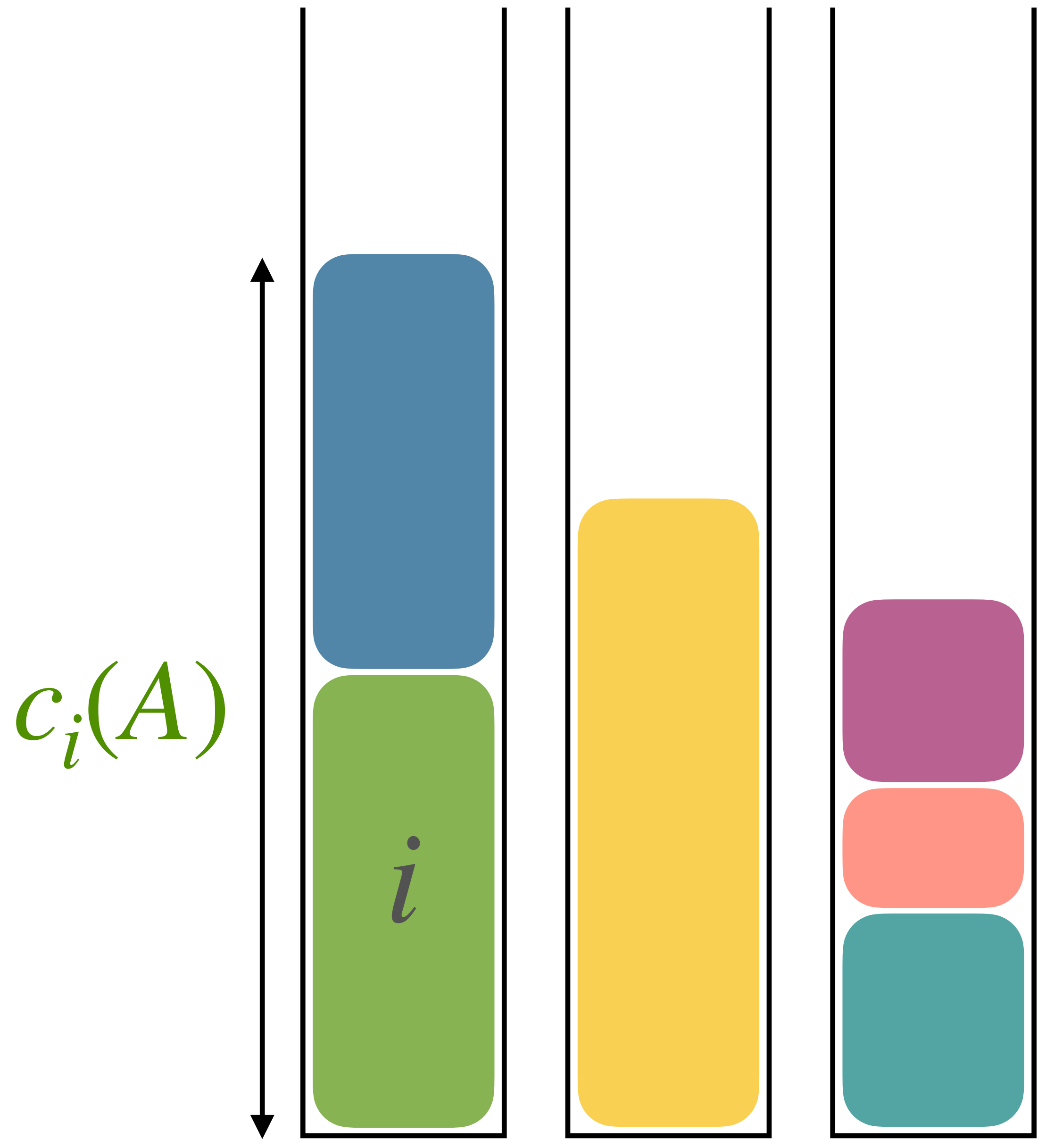

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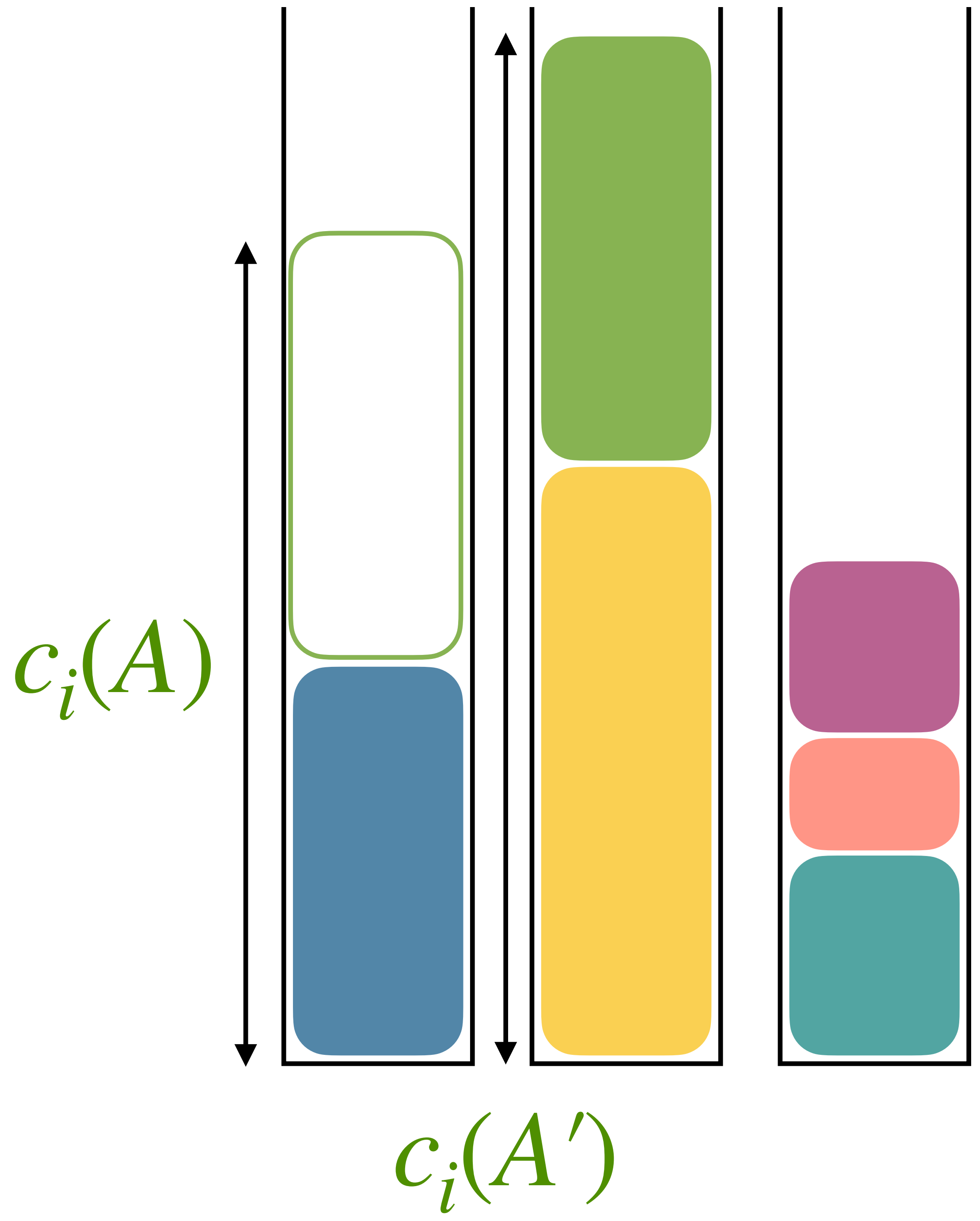

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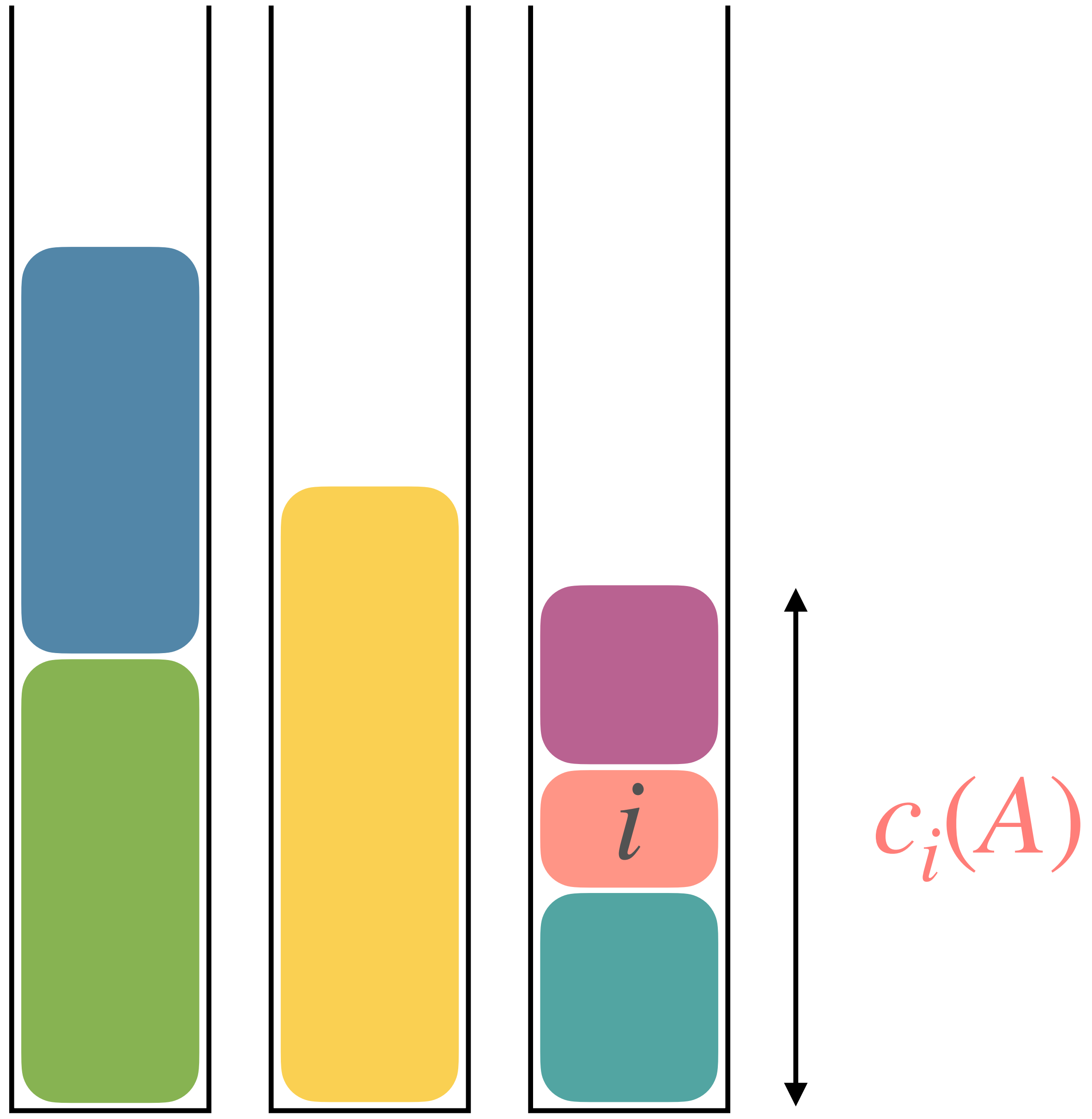

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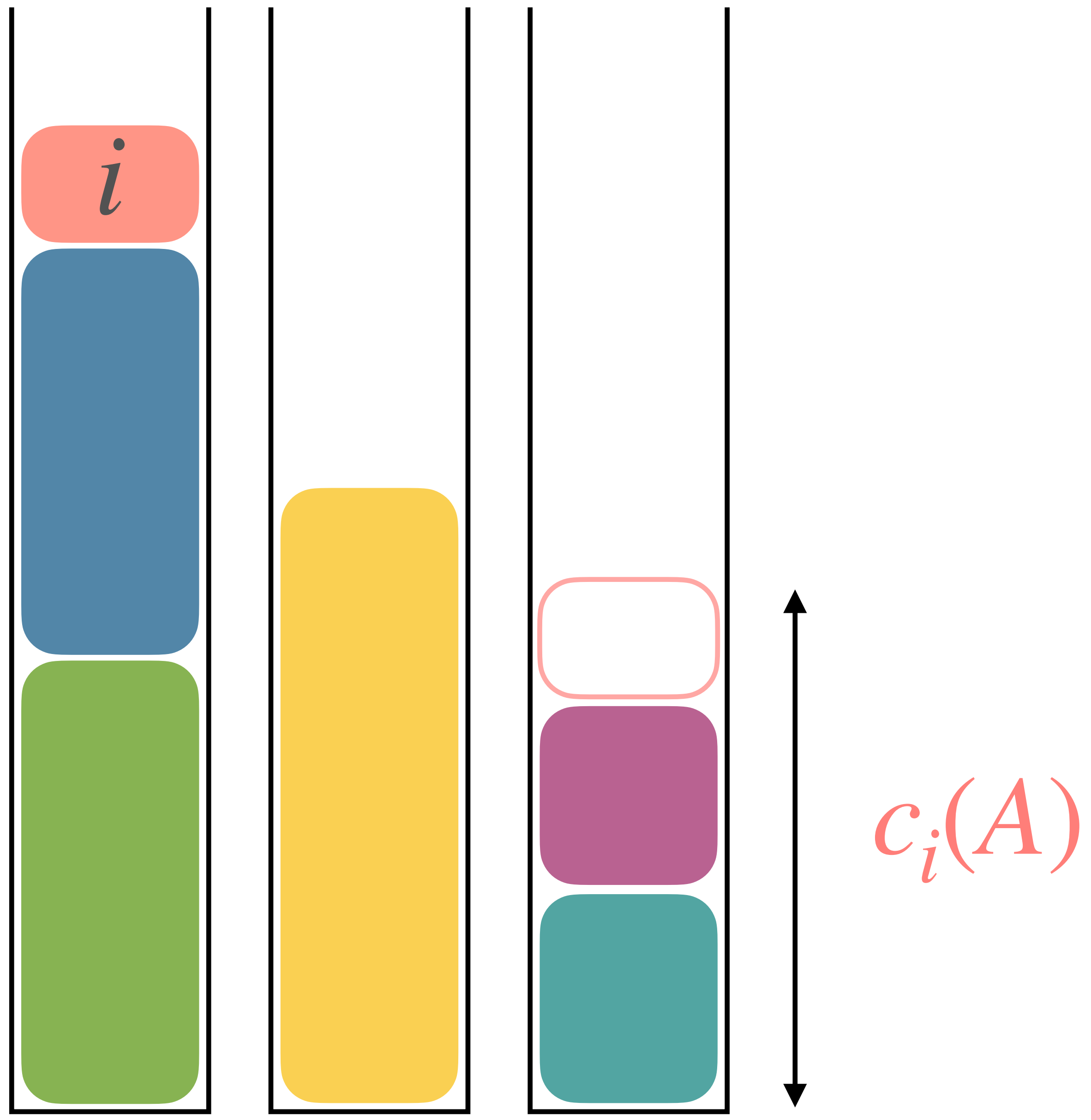

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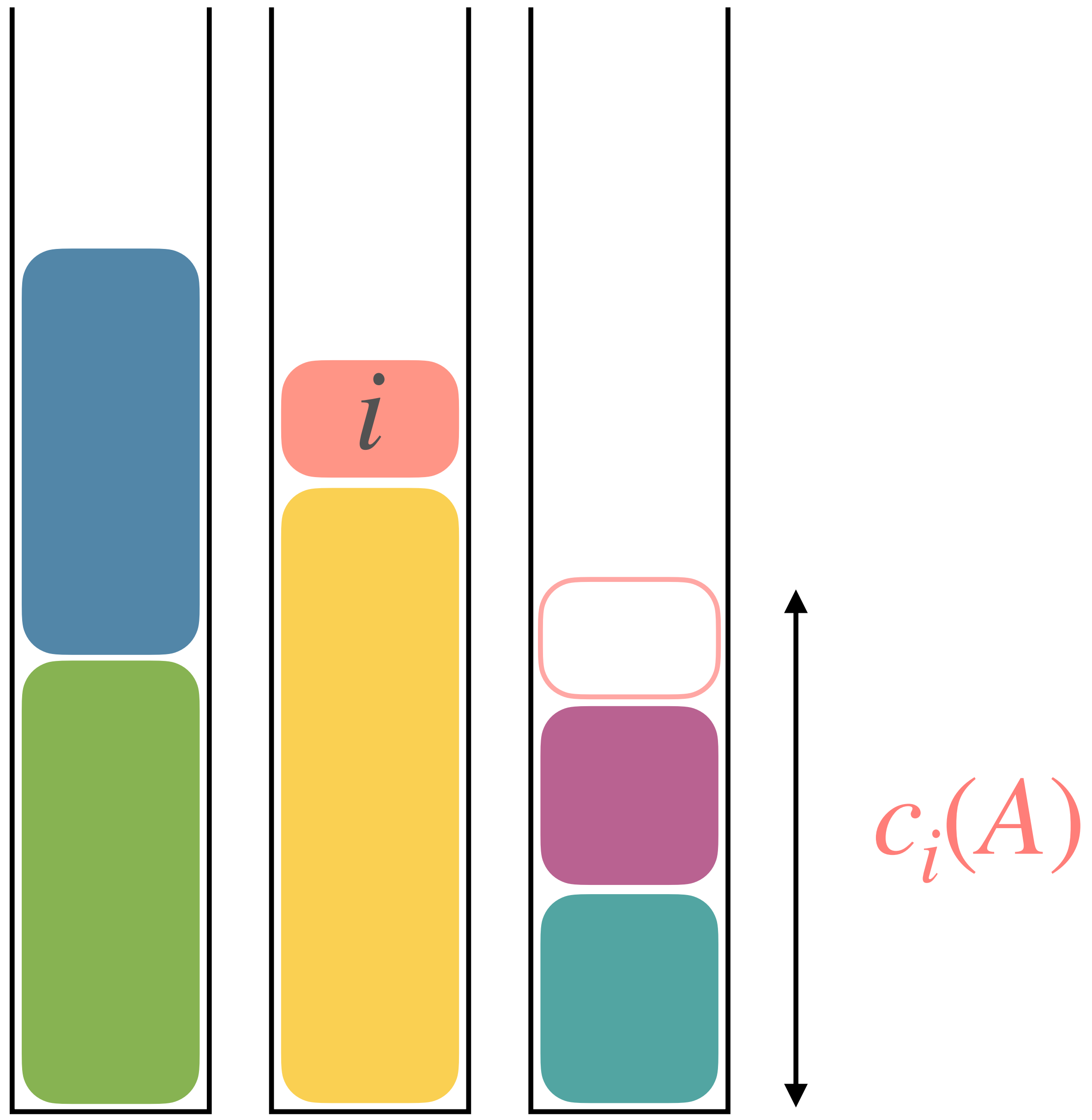


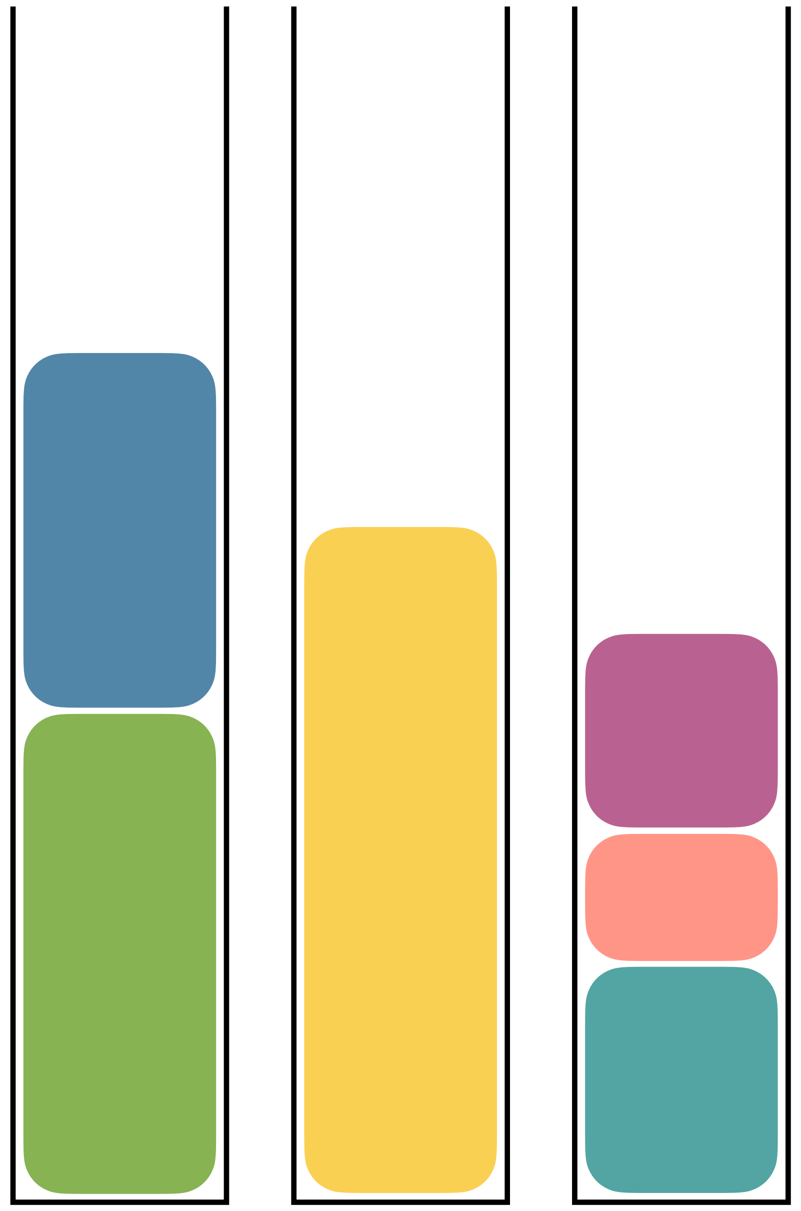
PoA of the Load Balancing Game

- Consider any instance of the load balancing game with n jobs of processing time $p_1, \cdots, p_n$ and m machines. Let A denote any Nash equilibrium assignment. Then, it holds that

$$\frac{cost(A)}{cost(\text{OPT})} \leq 2 - \frac{2}{m+1}$$

PoA of the Load Balancing Game

- Let $\hat{k}$ be the machine with the highest load under assignment A and $\hat{i}$ is the smallest job on $\hat{k}$



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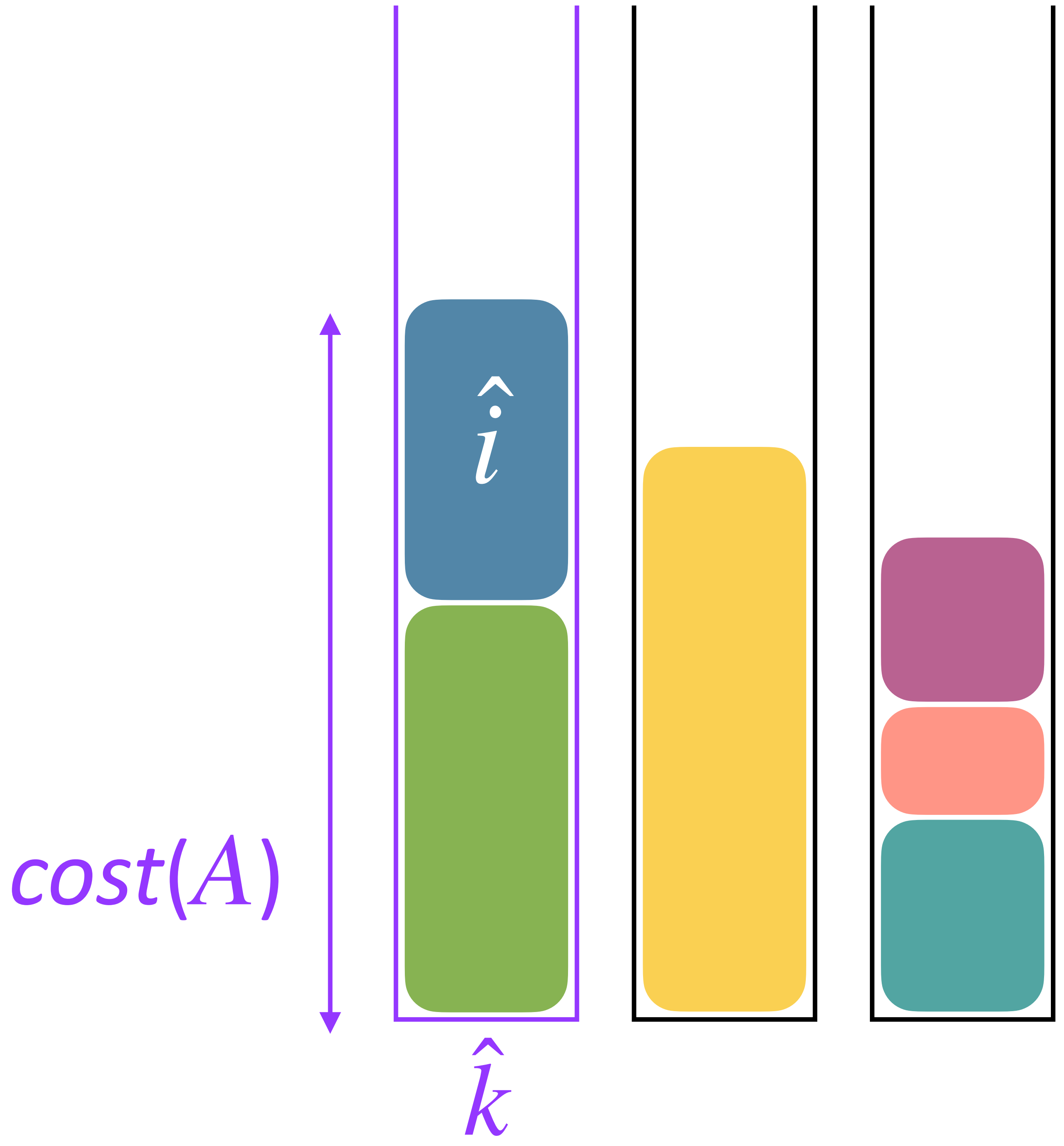

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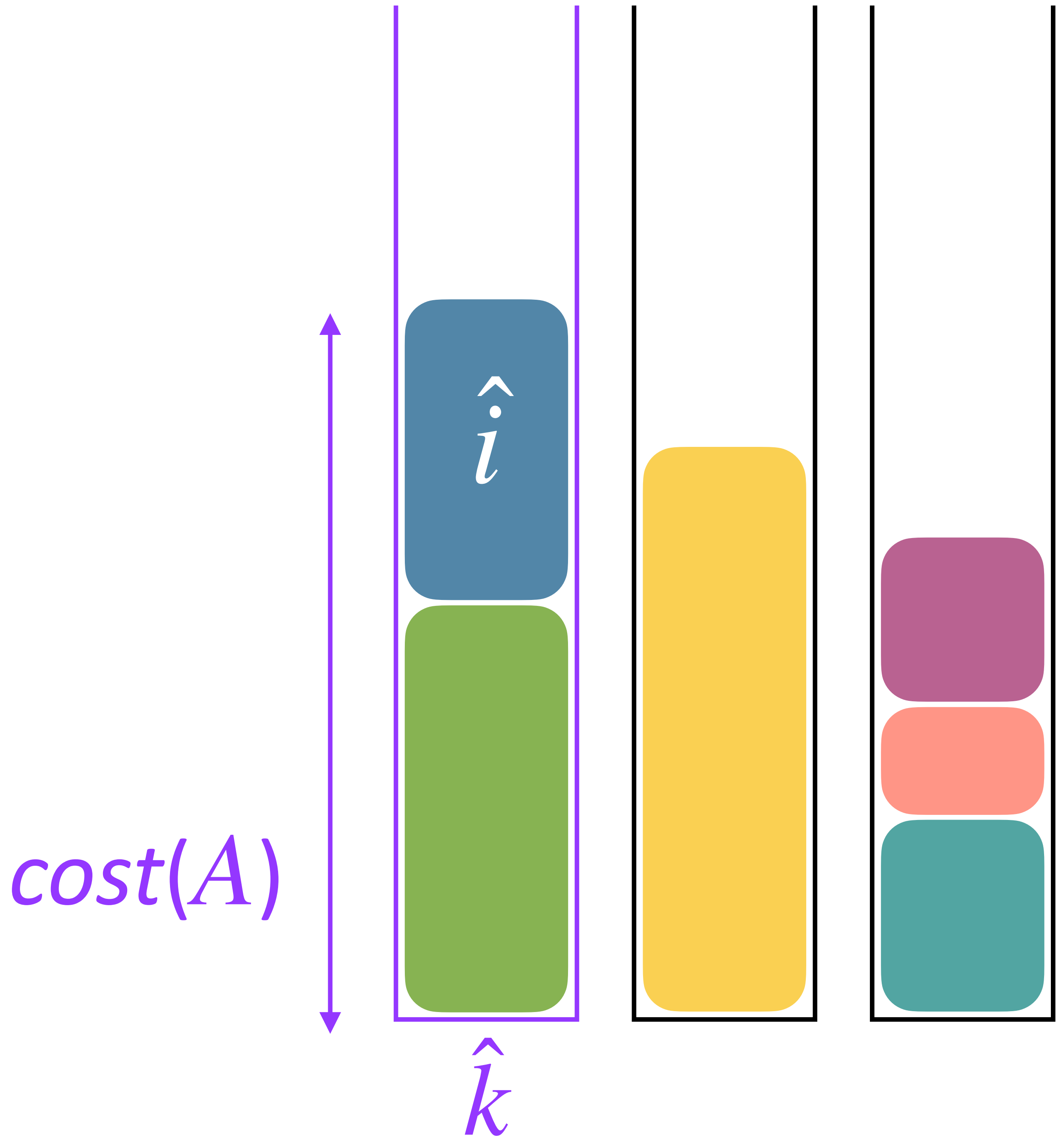
- Let $\hat{k}$ be the machine with the highest load under assignment A and $\hat{i}$ is the smallest job on $\hat{k}$
  - Without loss of generality, there are at least two tasks on $\hat{k}$ (otherwise, *cost*(OPT) $=$ *cost*(A) and the theorem is proven)

    $\Rightarrow p_{\hat{i}} \leq \dfrac{cost(A)}{2}$



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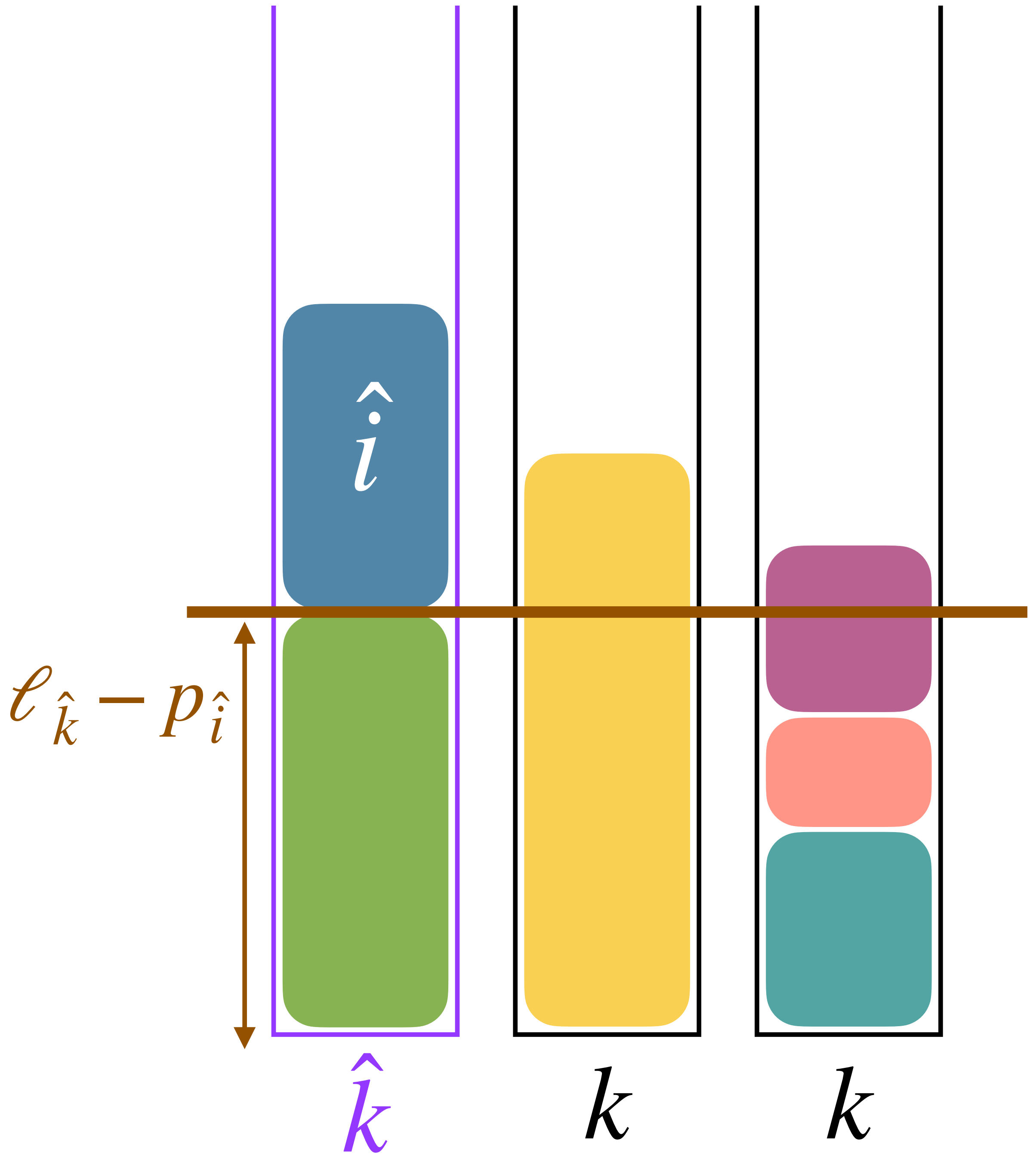
  - Suppose there is a machine $k \neq \hat{k}$ with load less than $\ell_{\hat{k}} - p_{\hat{i}}$. Then, moving job $\hat{i}$ from $\hat{k}$ to k would decrease the cost of the agent



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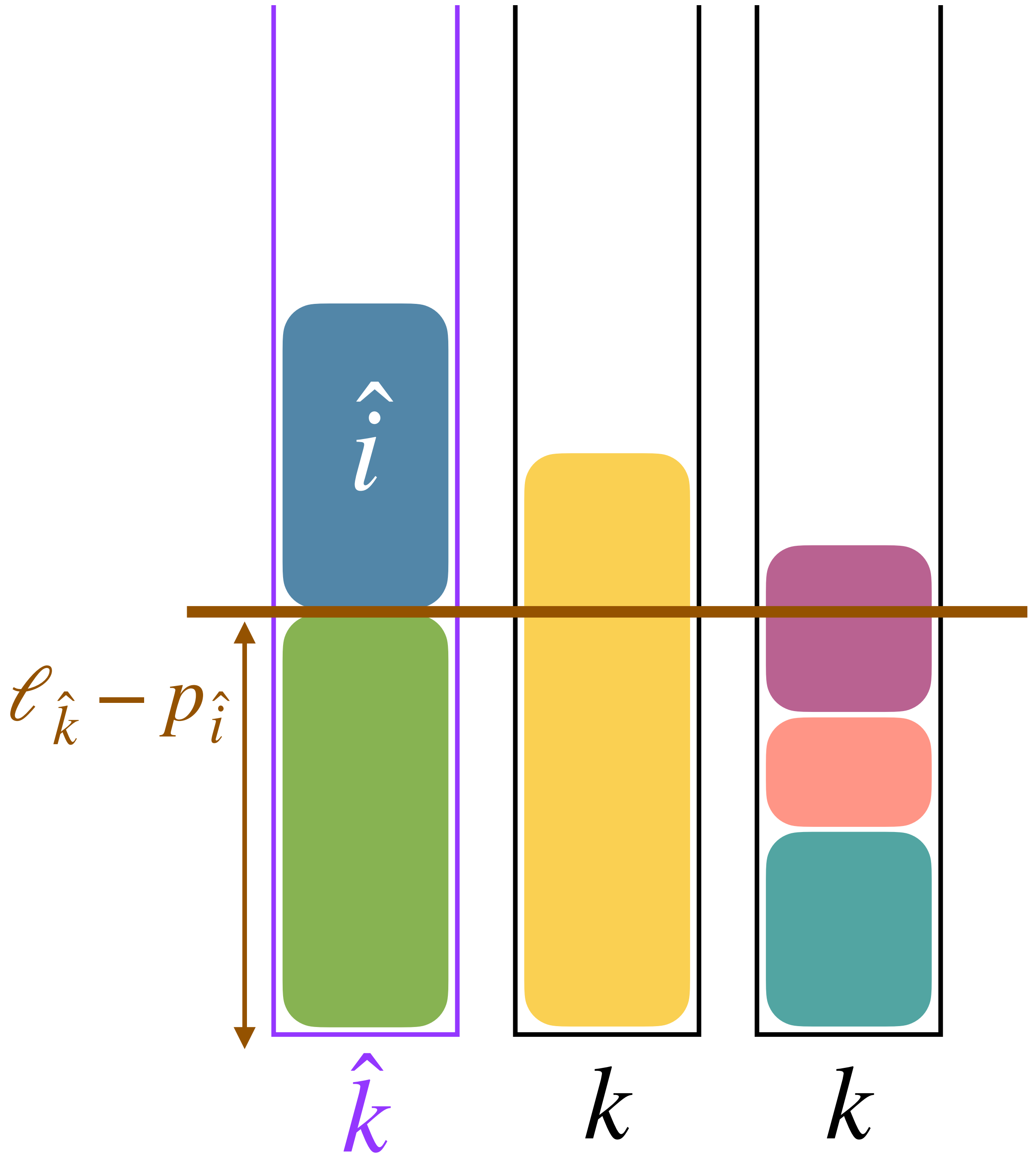
    - Since A is a Nash equilibrium, this cannot happen



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  - Without loss of generality, there are at least two tasks on $\hat{k}$ (otherwise, *cost*(OPT) $=$ *cost*(A) and the theorem is proven)

    $\Rightarrow p_{\hat{i}} \leq \dfrac{cost(A)}{2}$
  - Suppose there is a machine $k \neq \hat{k}$ with load less than $\ell_{\hat{k}} - p_{\hat{i}}$. Then, moving job $\hat{i}$ from $\hat{k}$ to k would decrease the cost of the agent
    - Since A is a Nash equilibrium, this cannot happen

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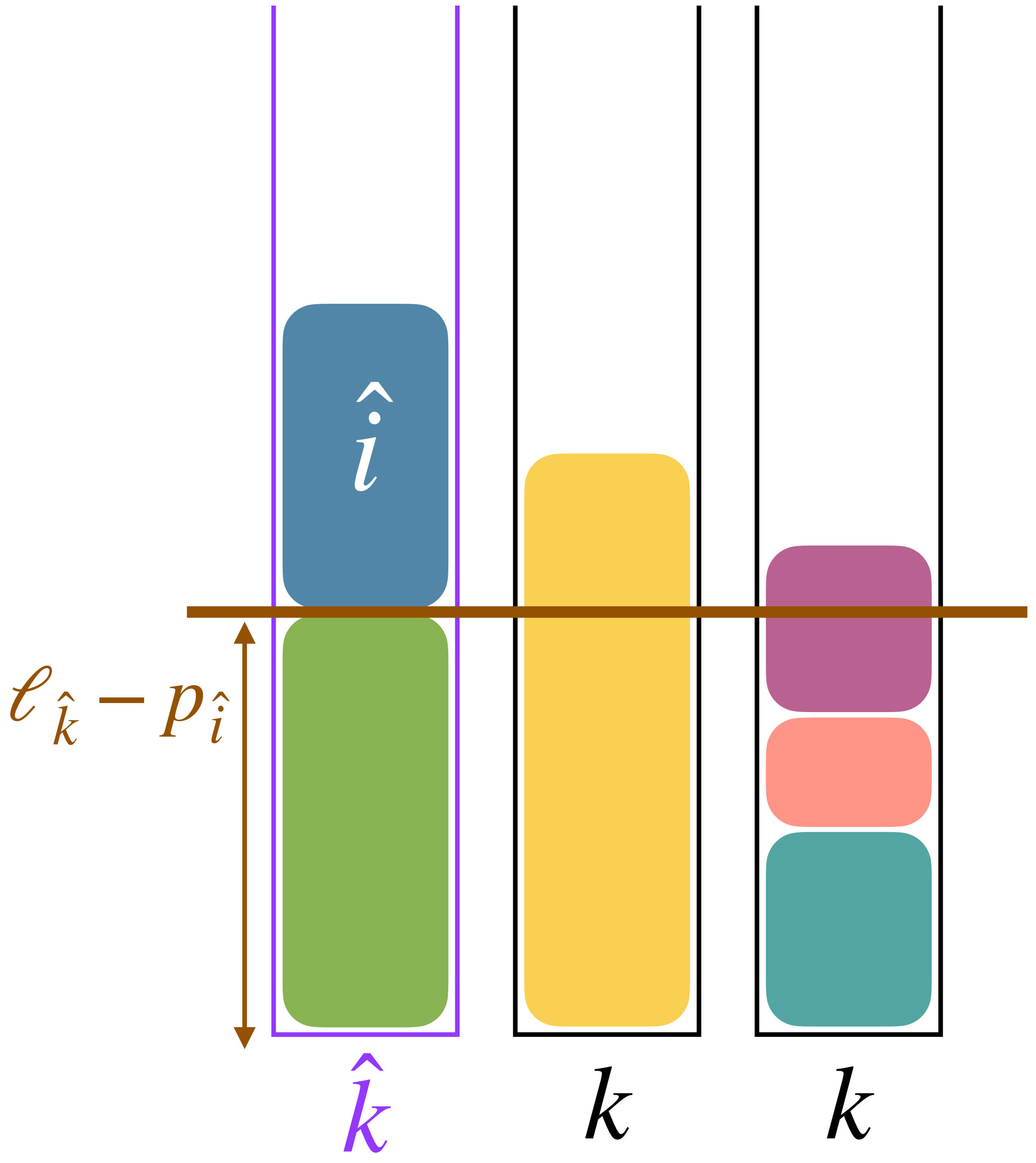

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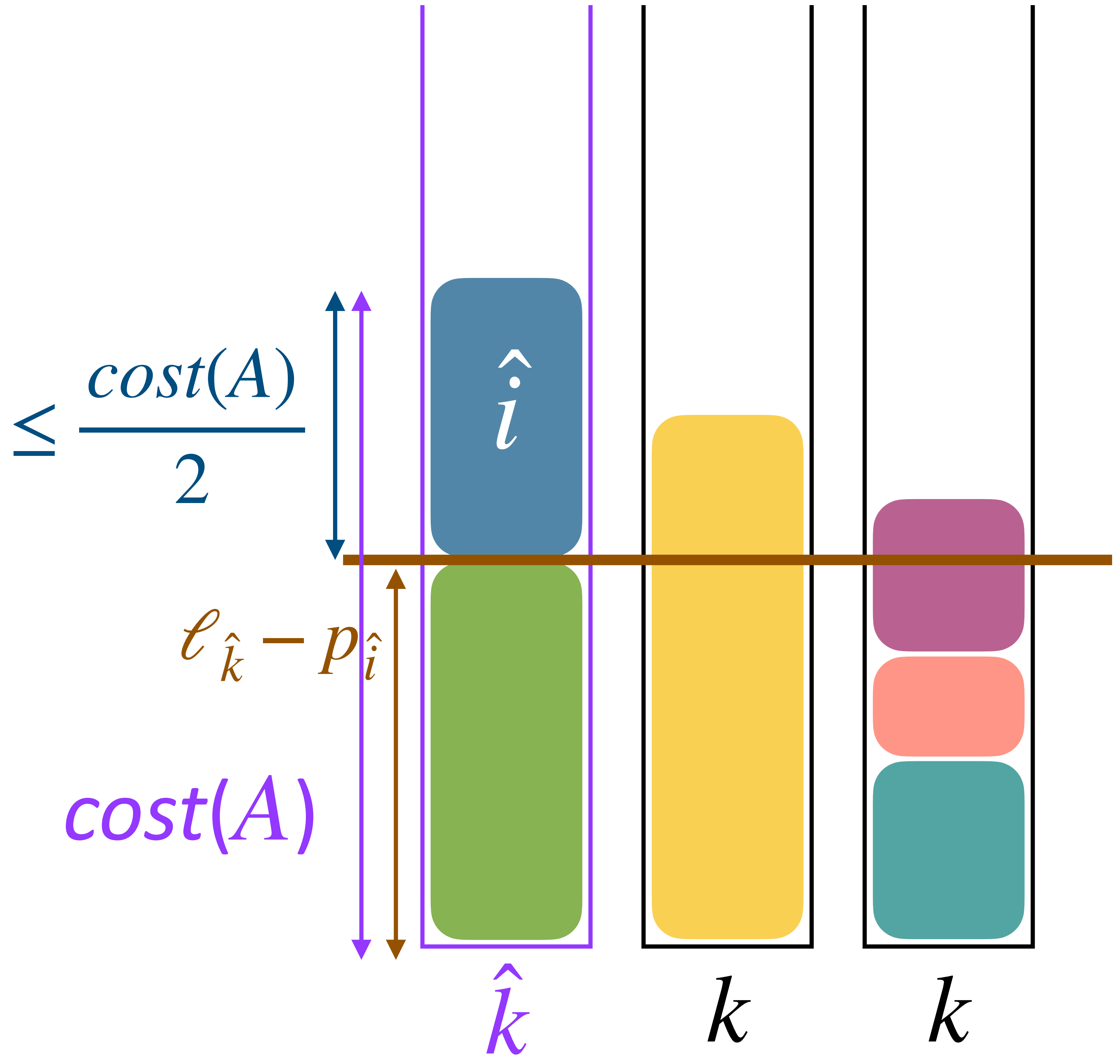

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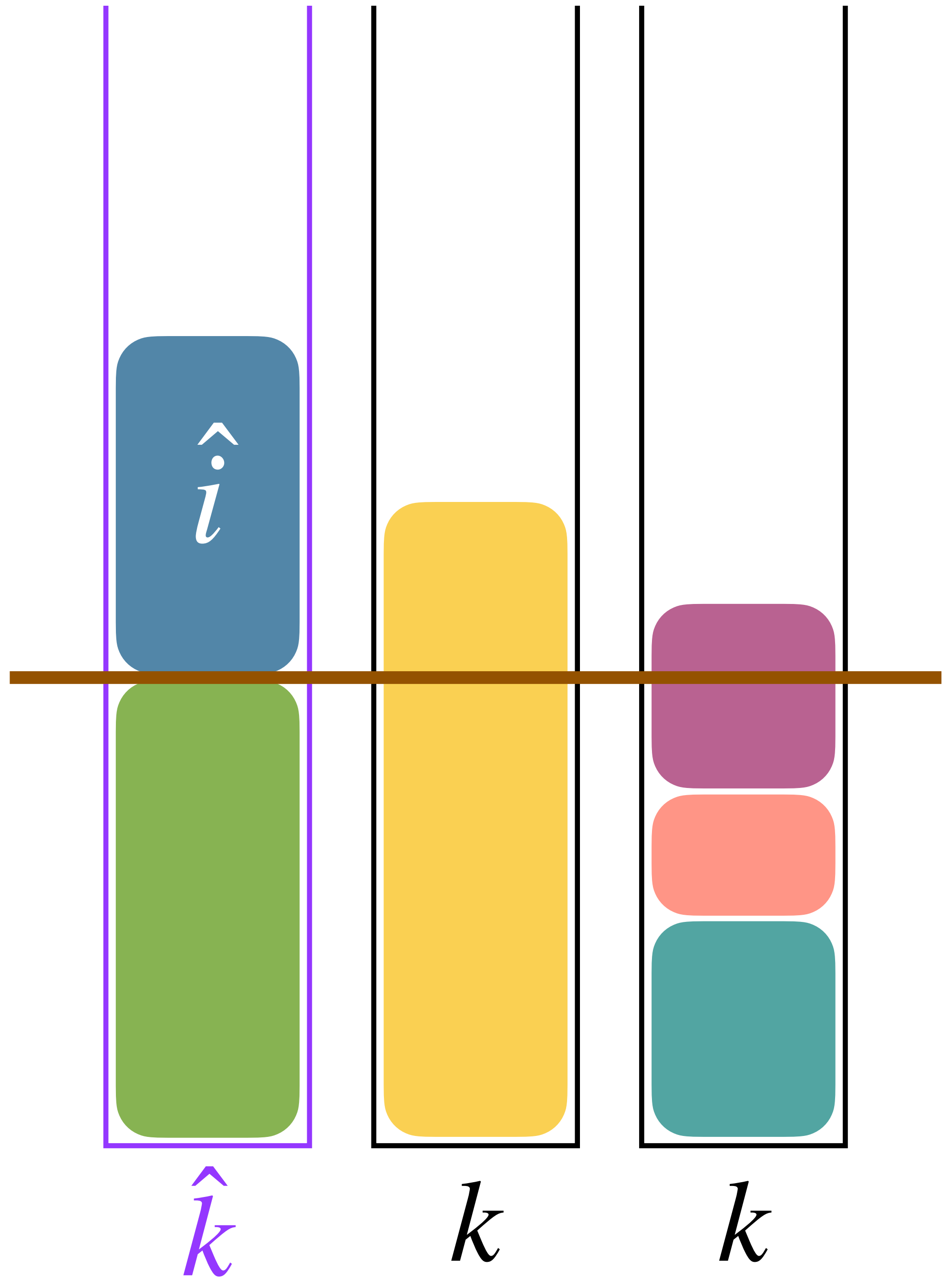

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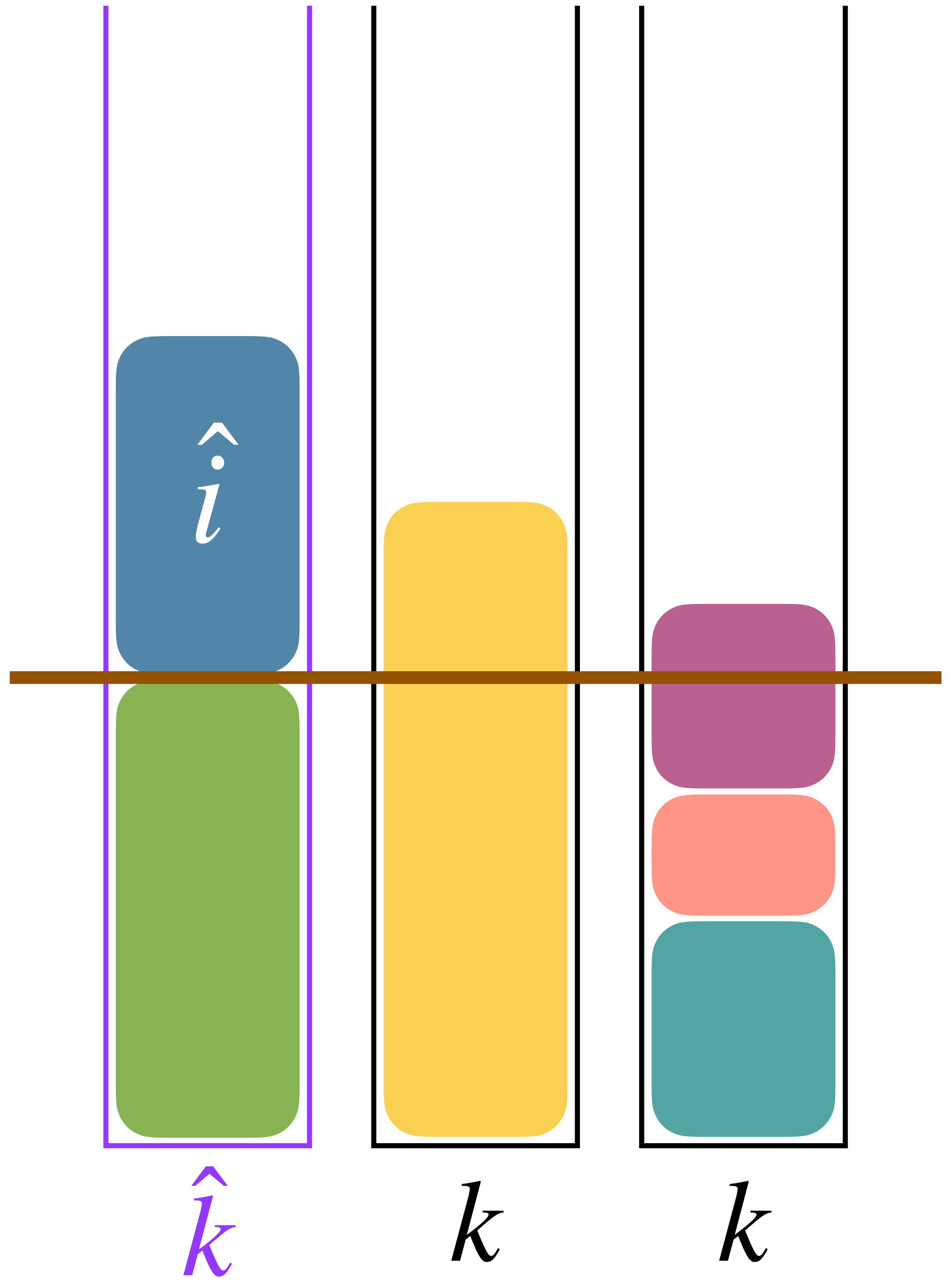
    $$\Rightarrow \frac{cost(A)}{cost(\text{OPT})} \leq \frac{2m}{m+1} = 2 - \frac{2}{m+1}$$



What happened

- The PoA of the selfish load balancing game is at most $2 - \dfrac{2}{m+1}$

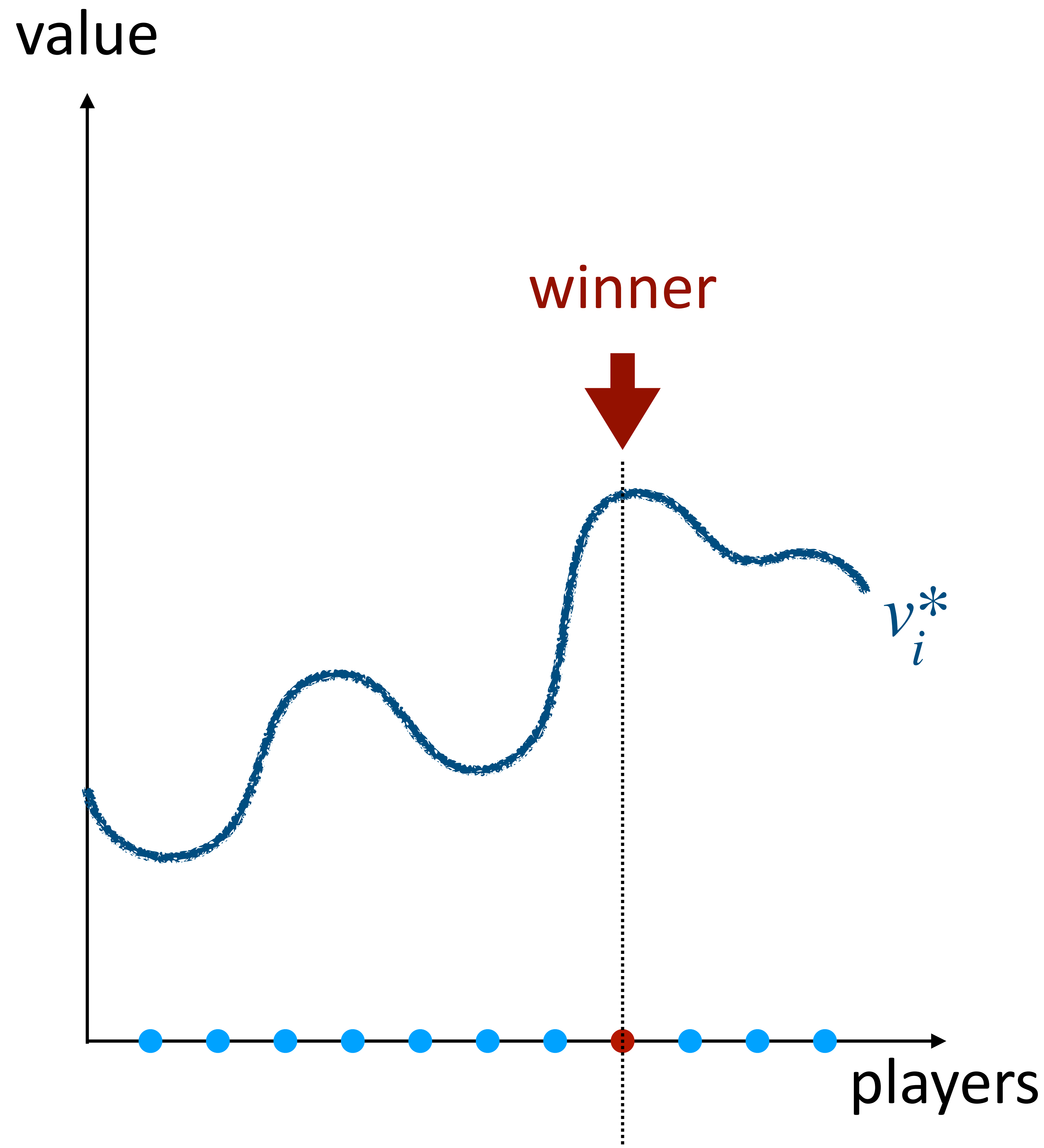
Outline

- Fundamental concepts
  - Game, players, strategies, payoffs/costs
- Nash Equilibrium
- Price of Anarchy
  - Selfish load balancing
- Mechanism design
  - Auction
  - Vickrey-Clarke-Groves mechanism

Auction Game

- Game: There is a valuable item.

- Each player (bidder) i has a value v_i^* for the good that he is “willing to pay” for the item and private to himself



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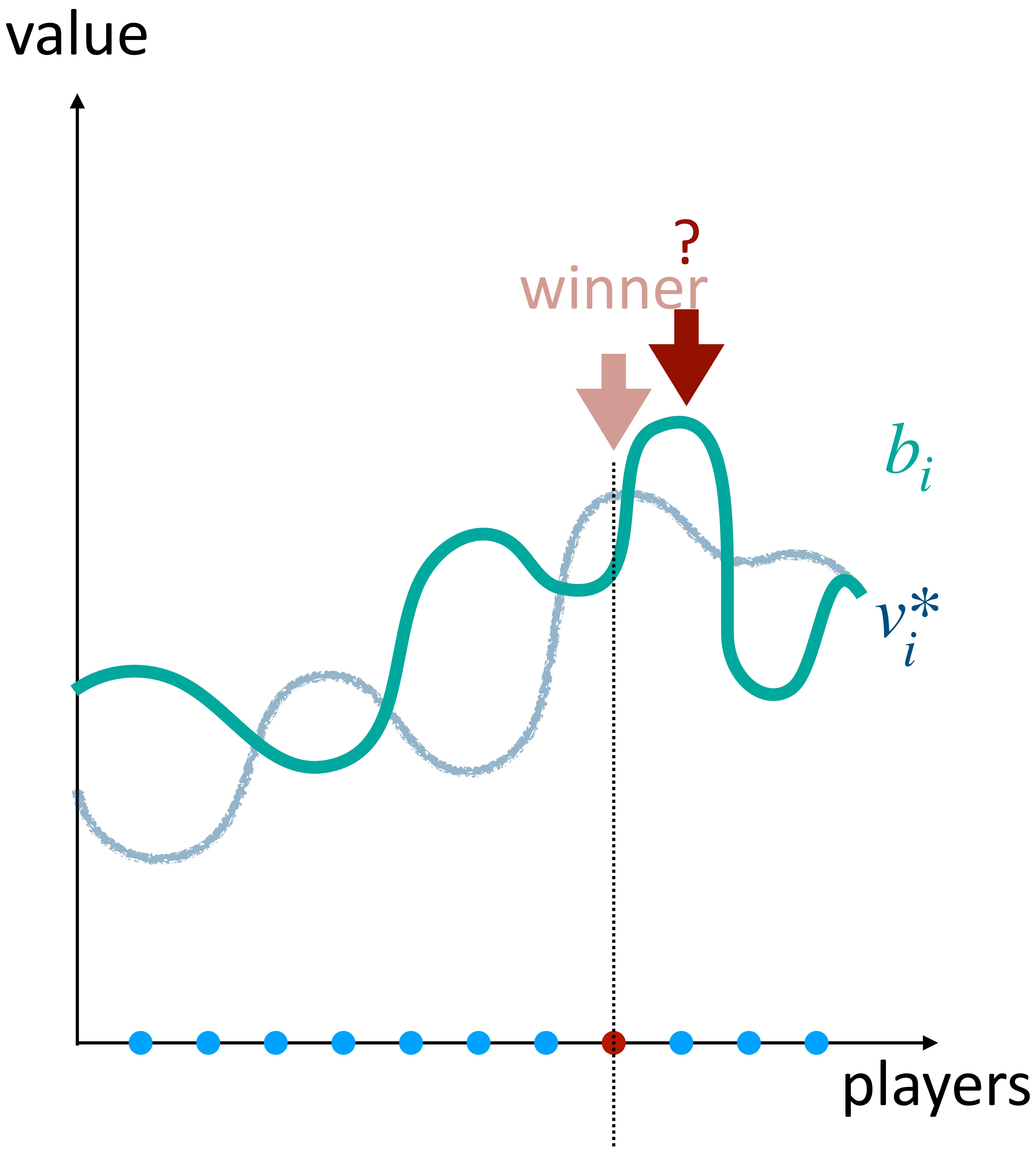

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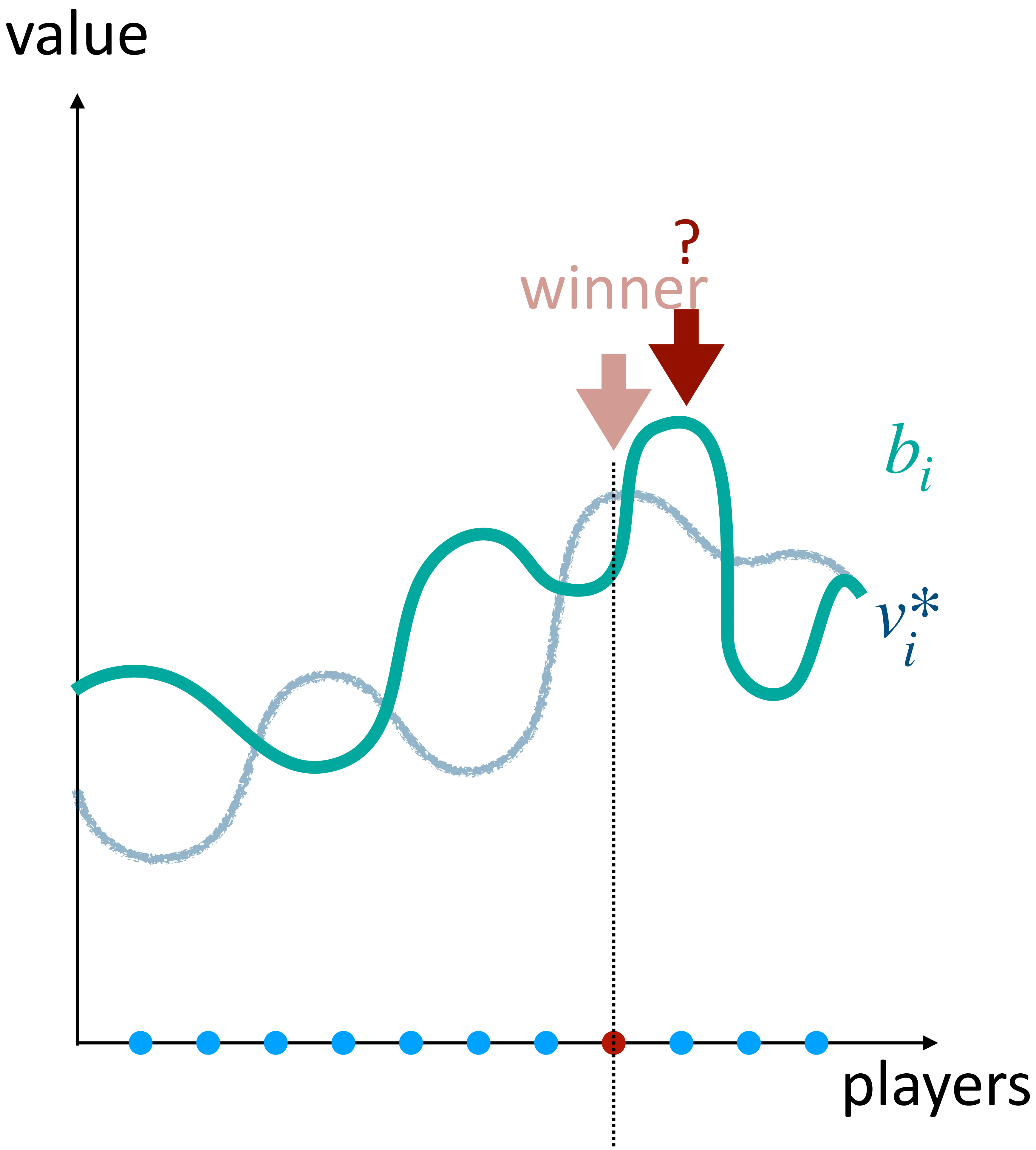

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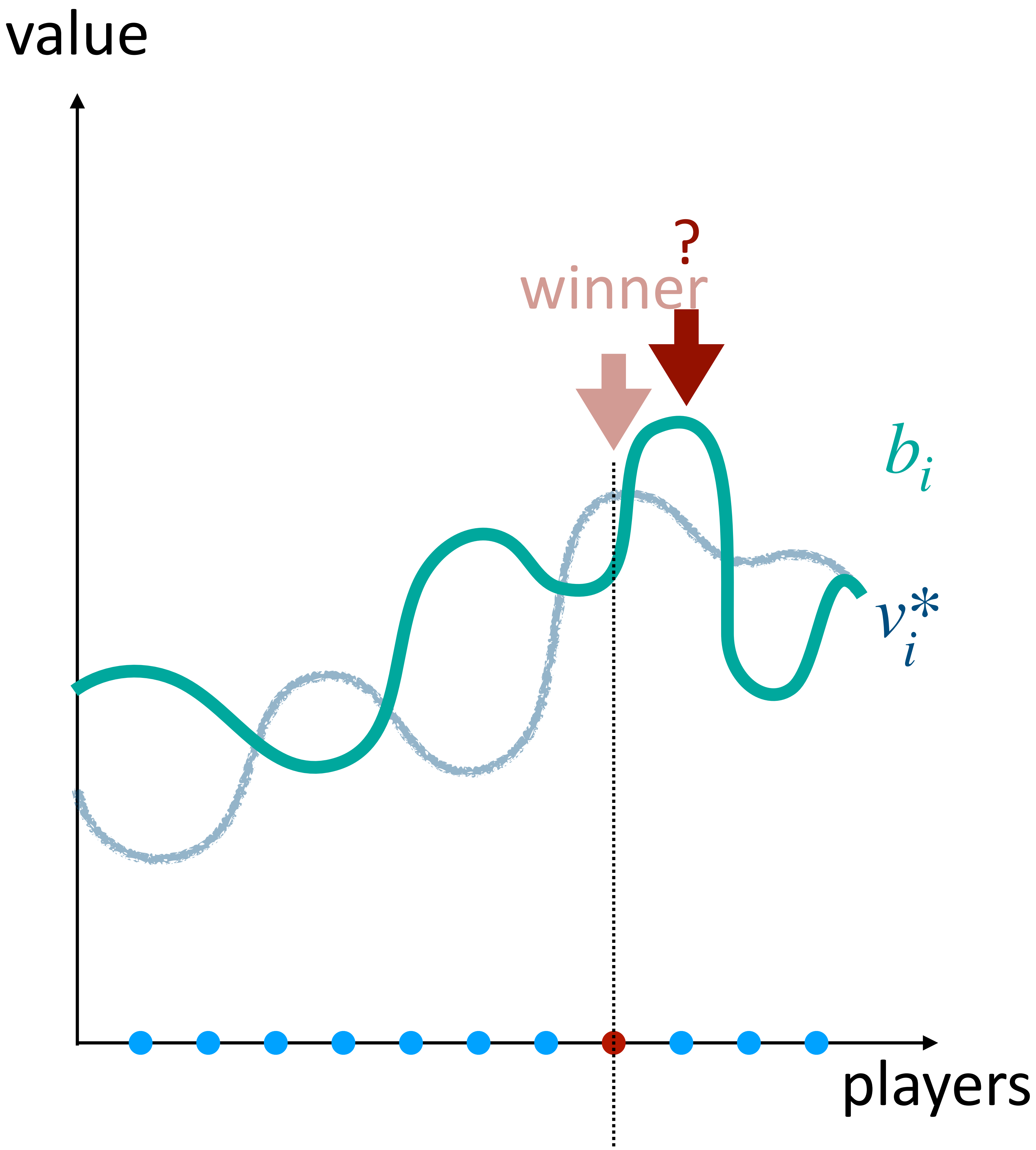
    - *Utility* of player i is 0 if he does not win, and $v_i^* - p$ if he wins at a price of p



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If player i wins, $u_i(f(\overrightarrow{b})) = v_i^* - p$

If player i loses, $u_i(f(\overrightarrow{b})) = 0$

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  - However, v_i^* is a private value, and the seller doesn't know them
    - Can players *strategically manipulate* the game?

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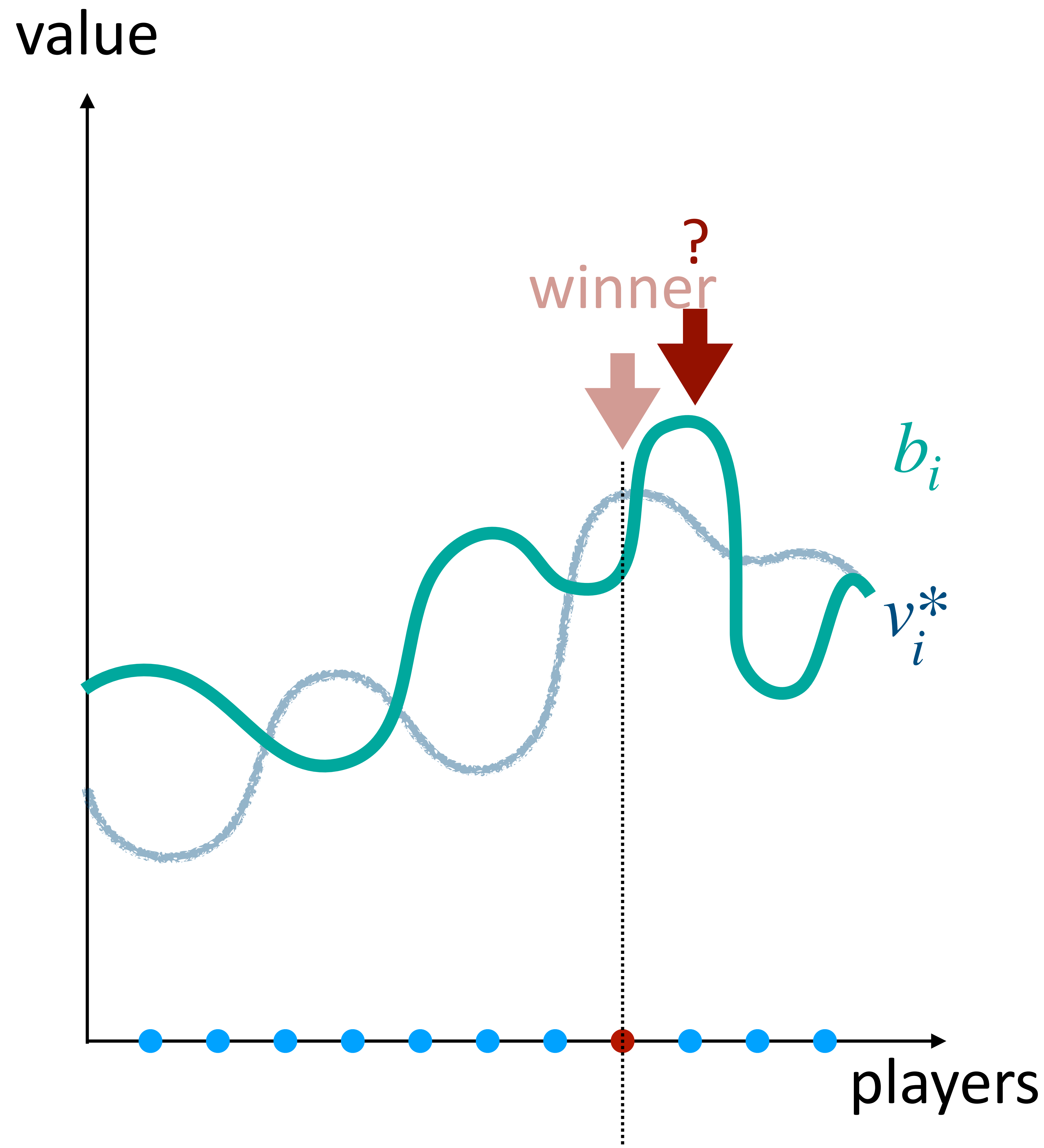

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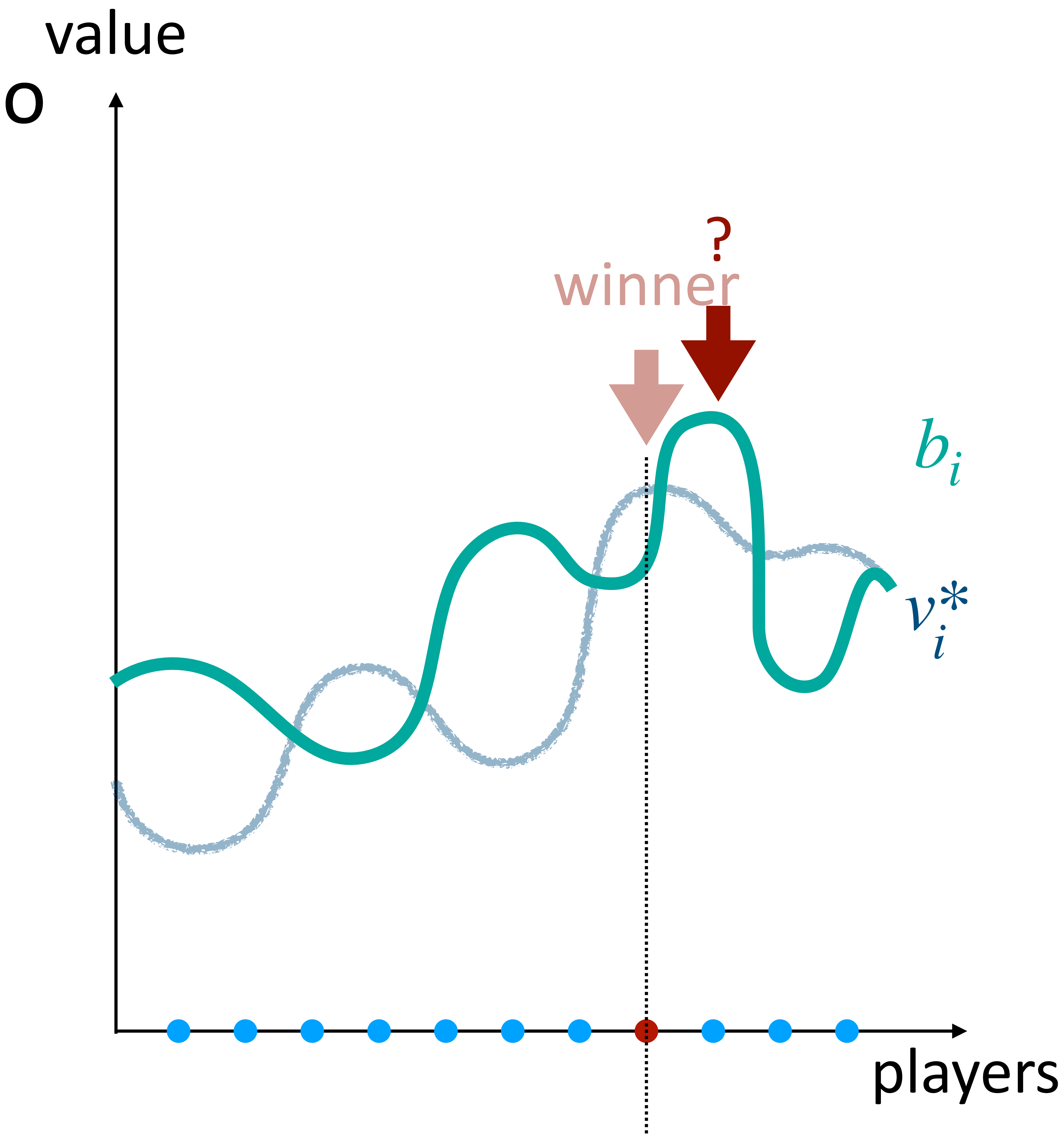
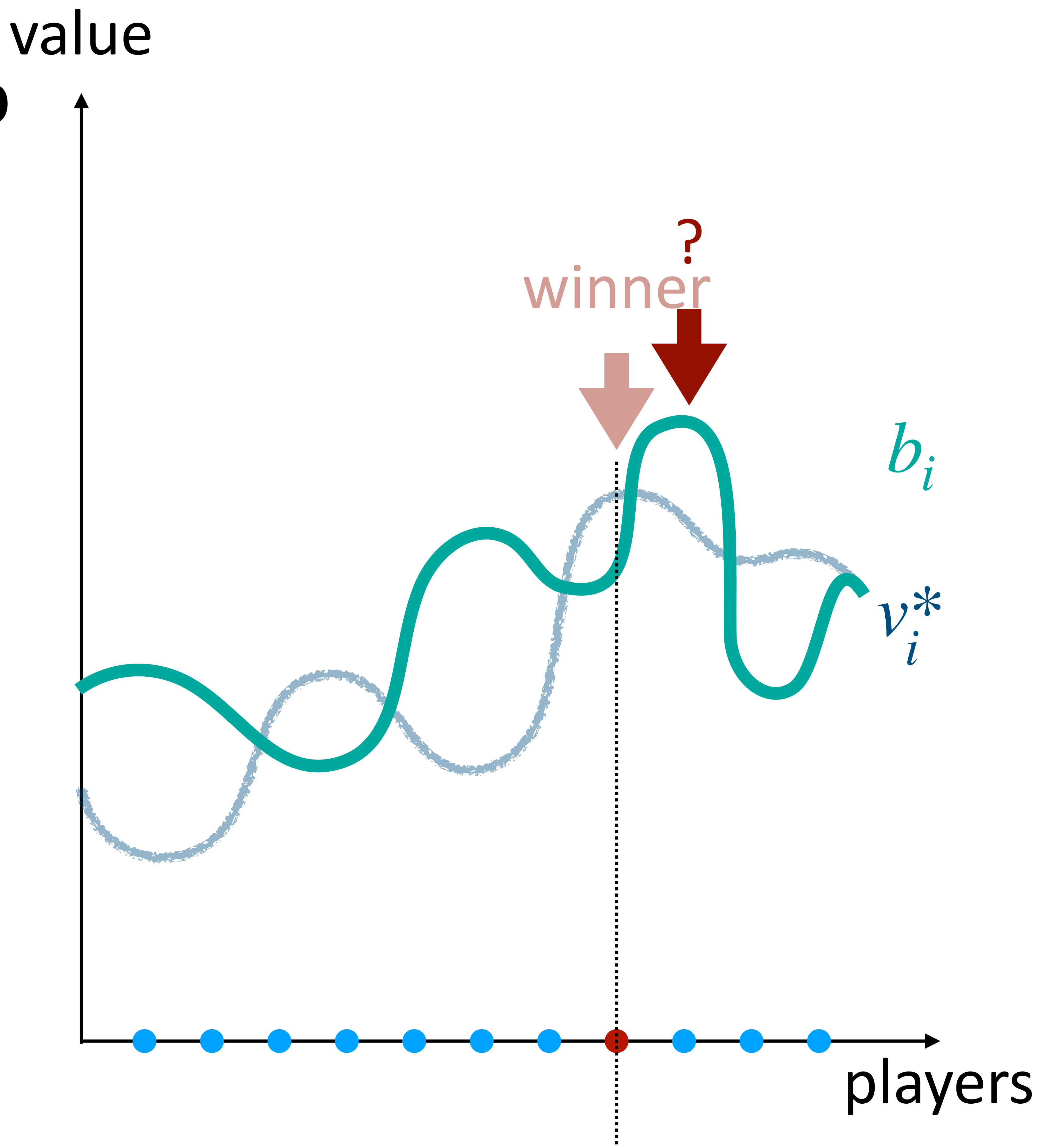


Truthfulness?

- No payment ($p = 0$): We give the item for free to the player with highest v_i^*
  - This method is easily manipulated: player can benefit by exaggerating his v_i^* by reporting bid $b_i \gg v_i^*$



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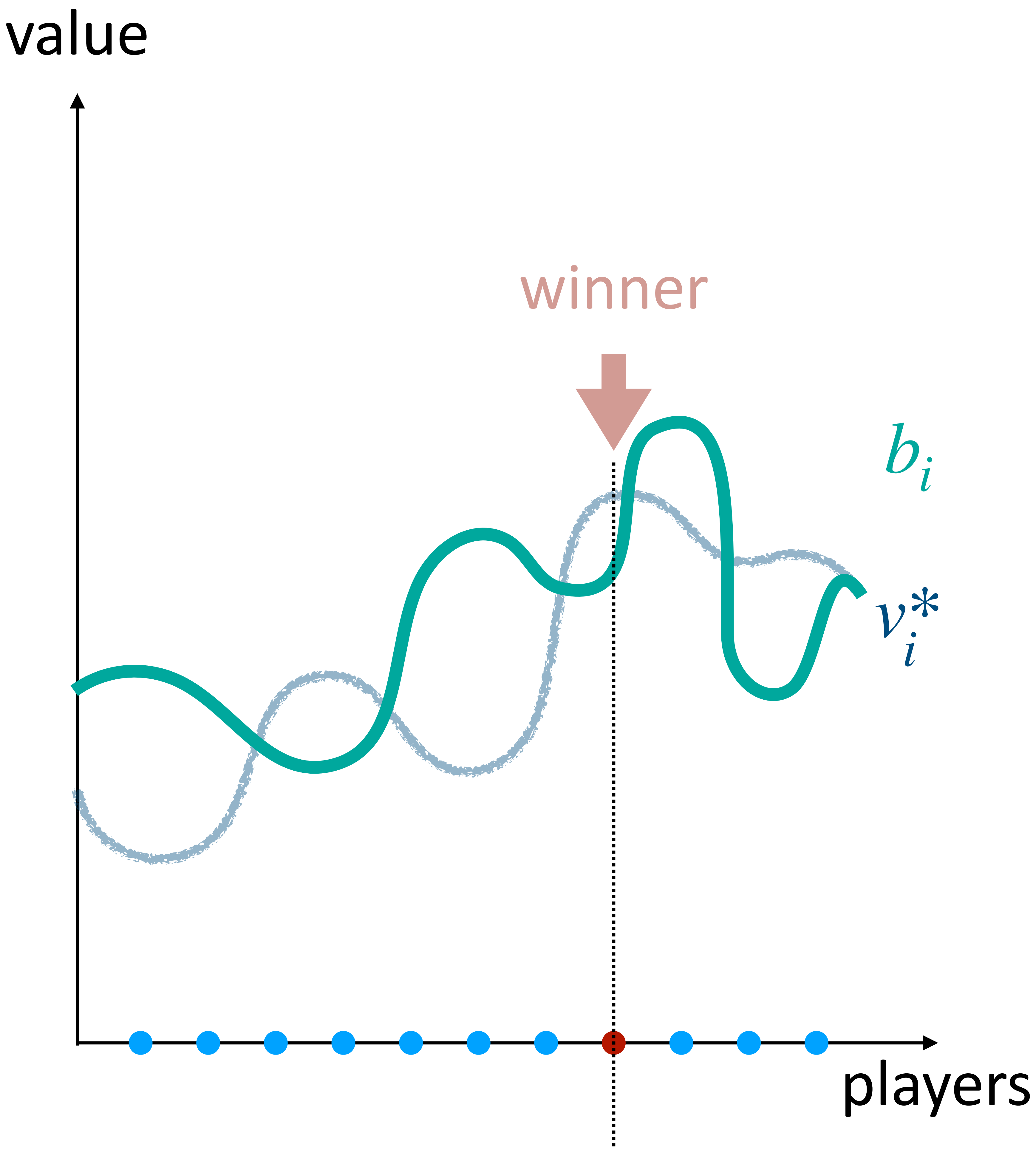

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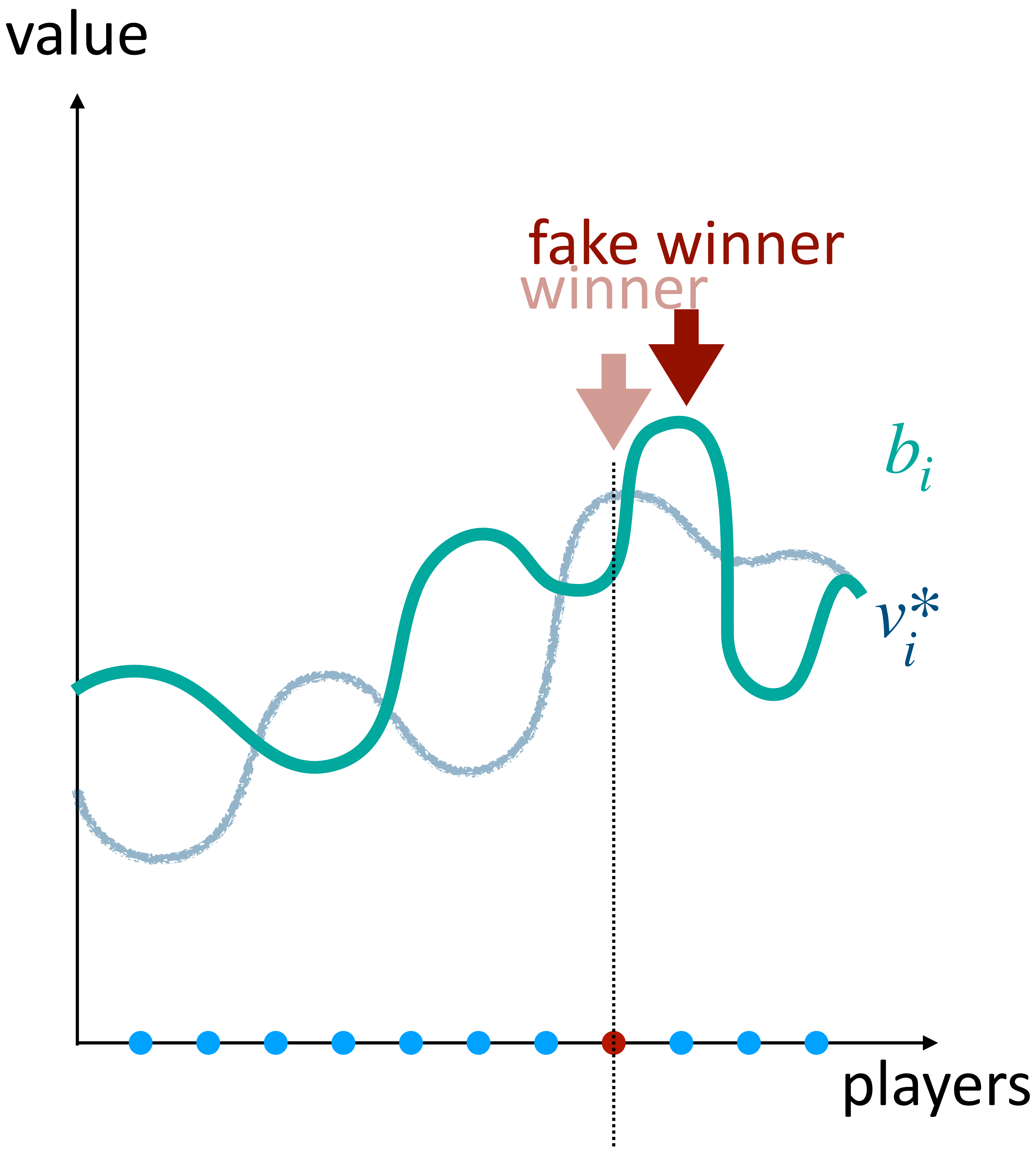

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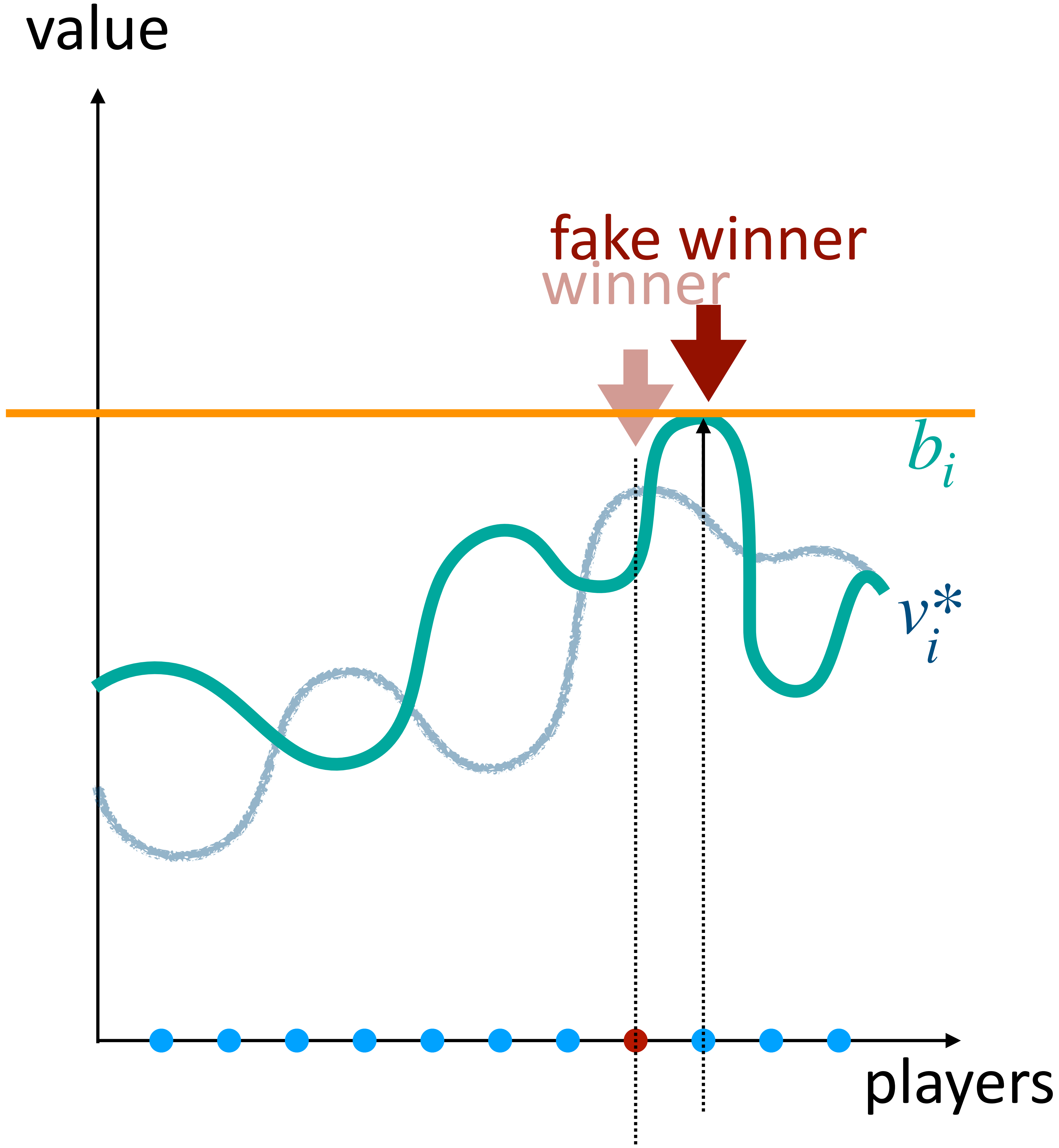

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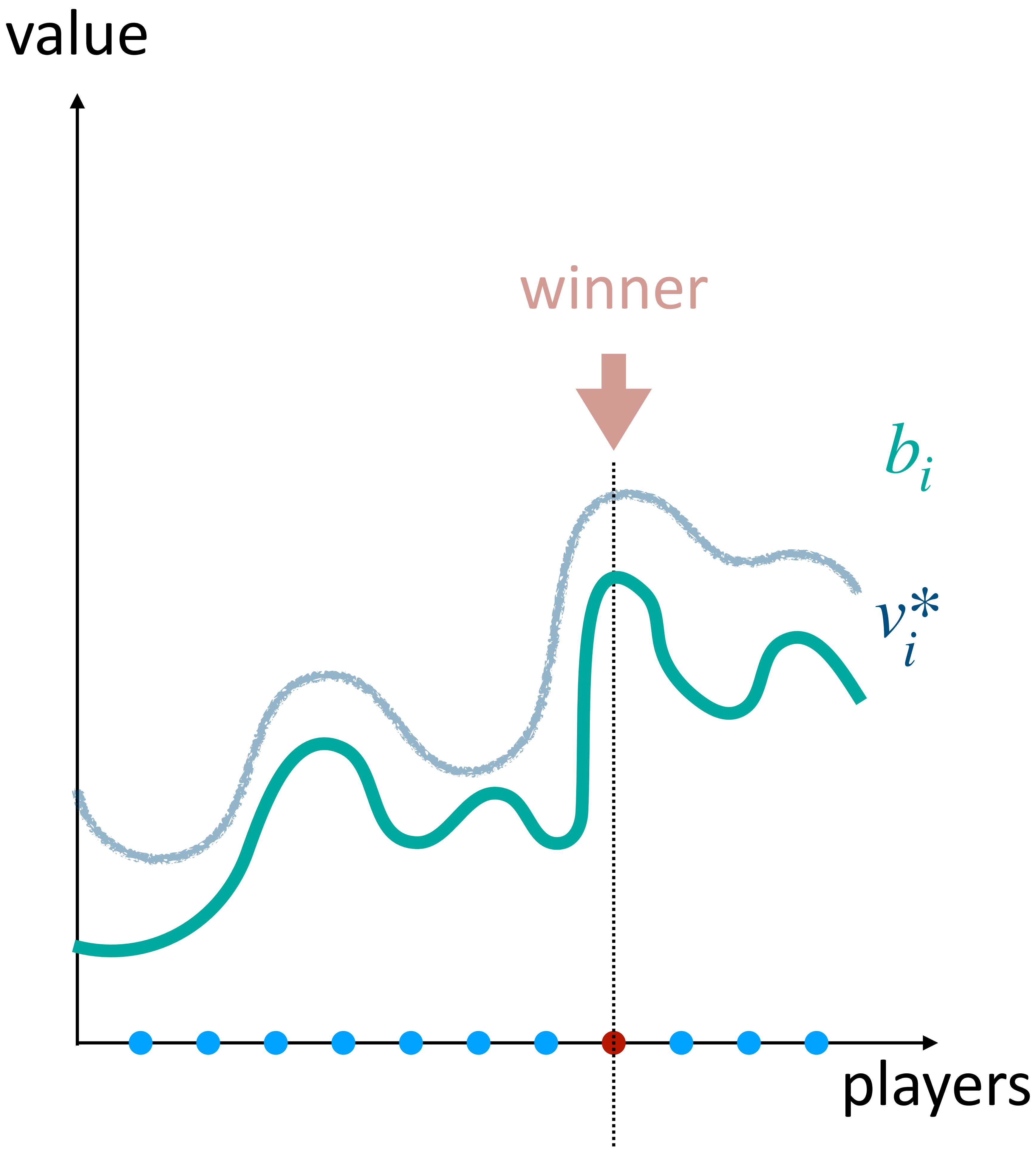

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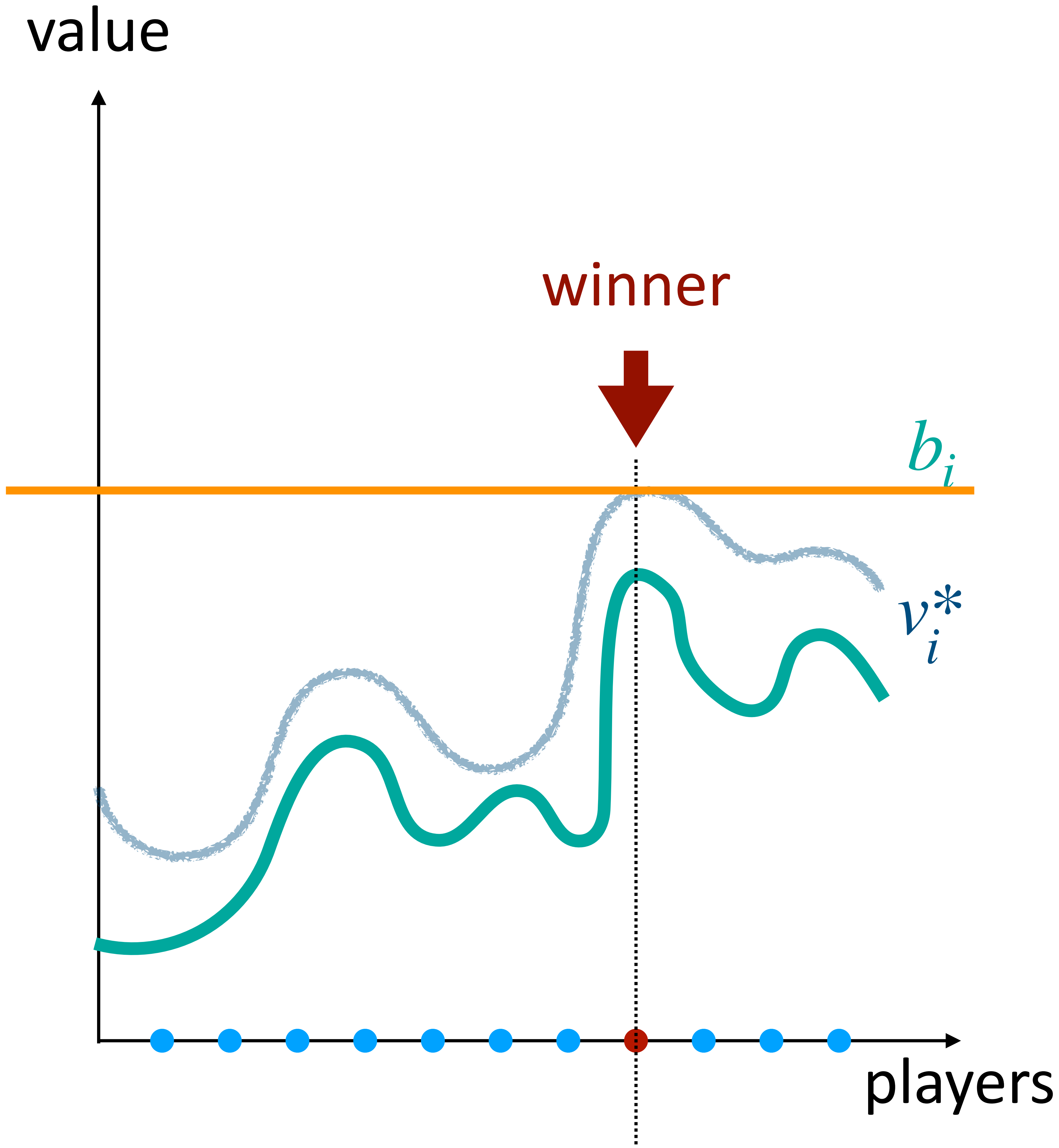

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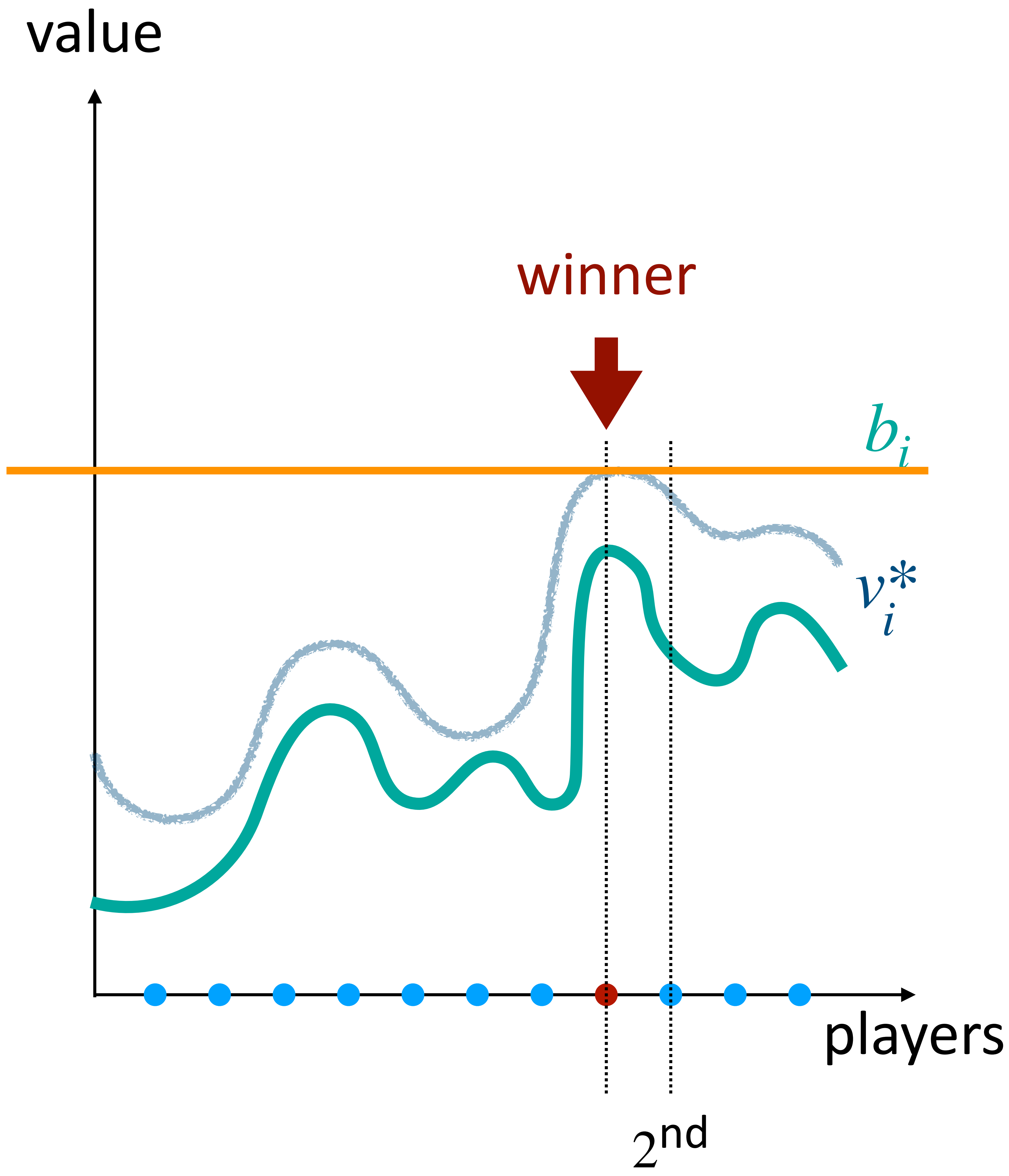

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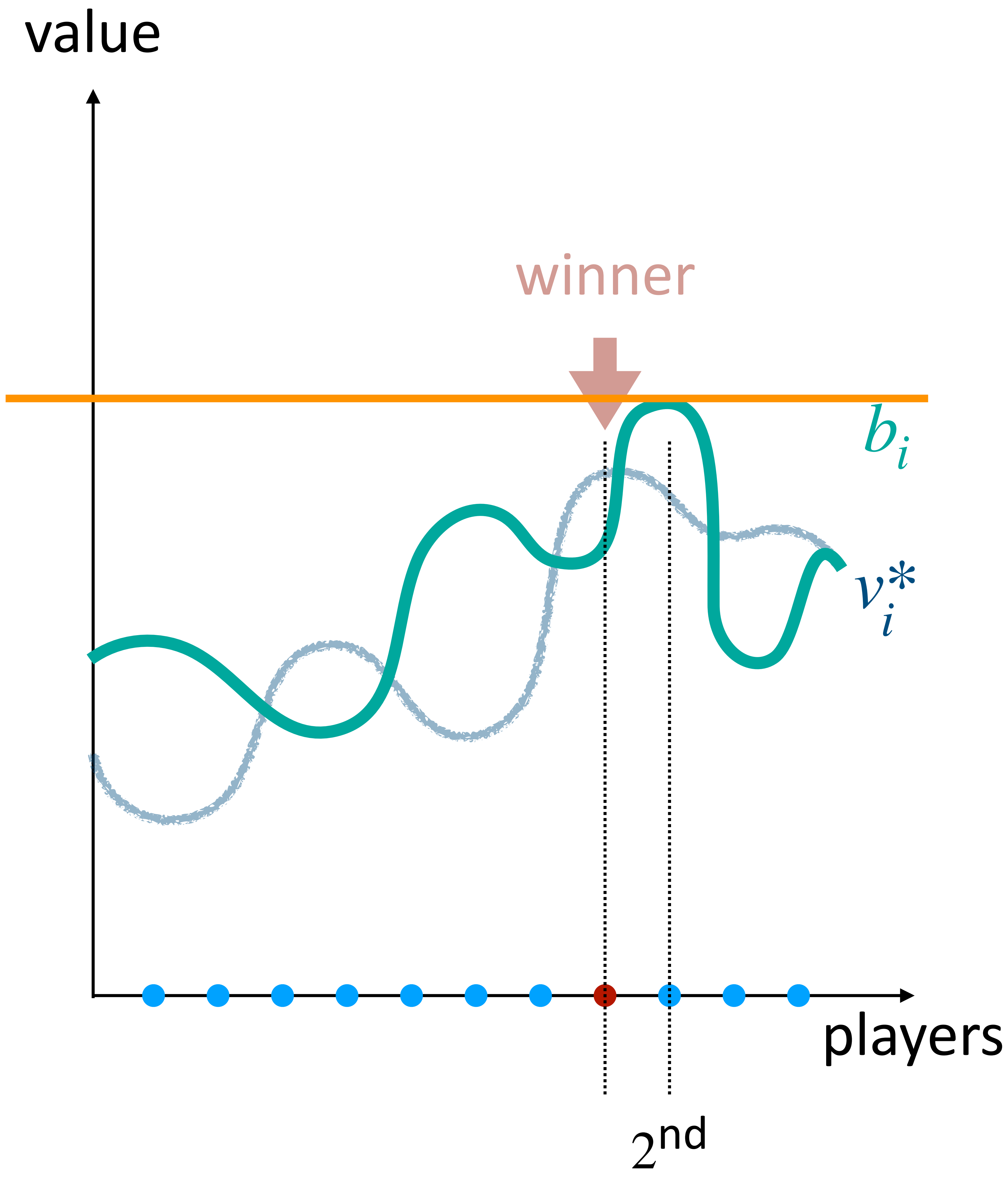
      - The better scenario is that he knows the second-highest bid and make b_w a bit larger than it



Vickrey's Second Price Auction

- Let the winner be the player i with the highest declared value of b_i, and let i pay the second highest declared bid
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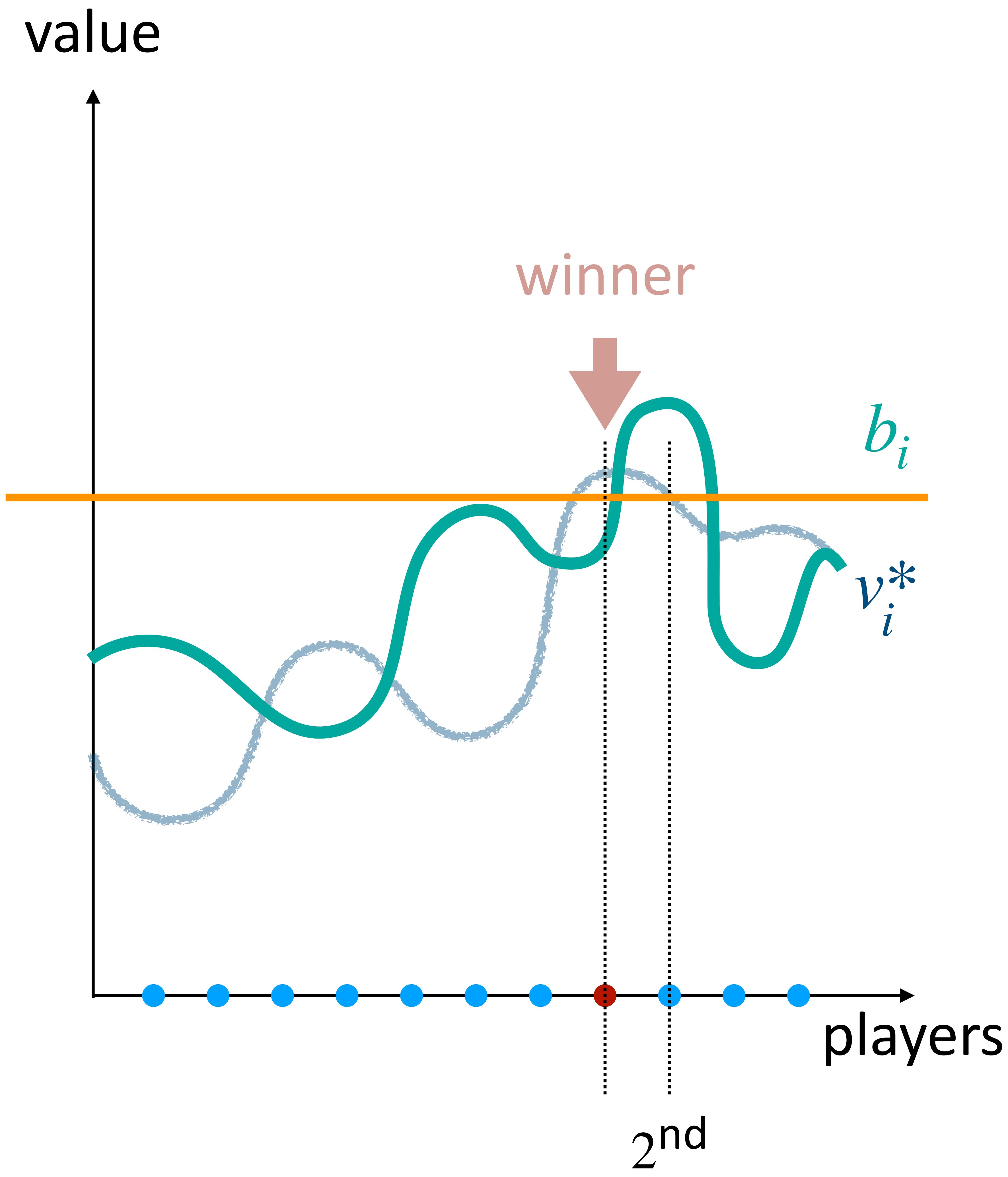

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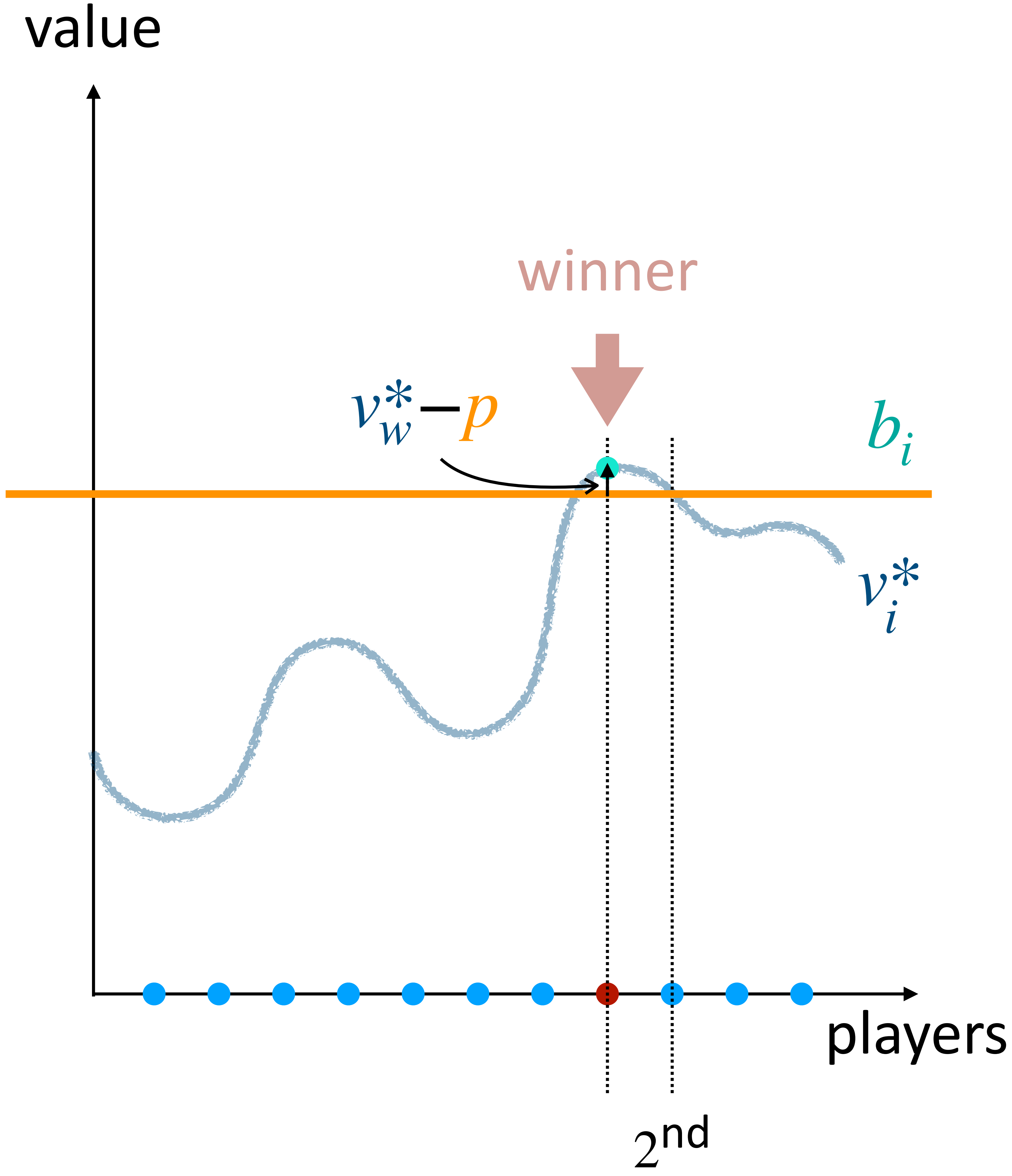

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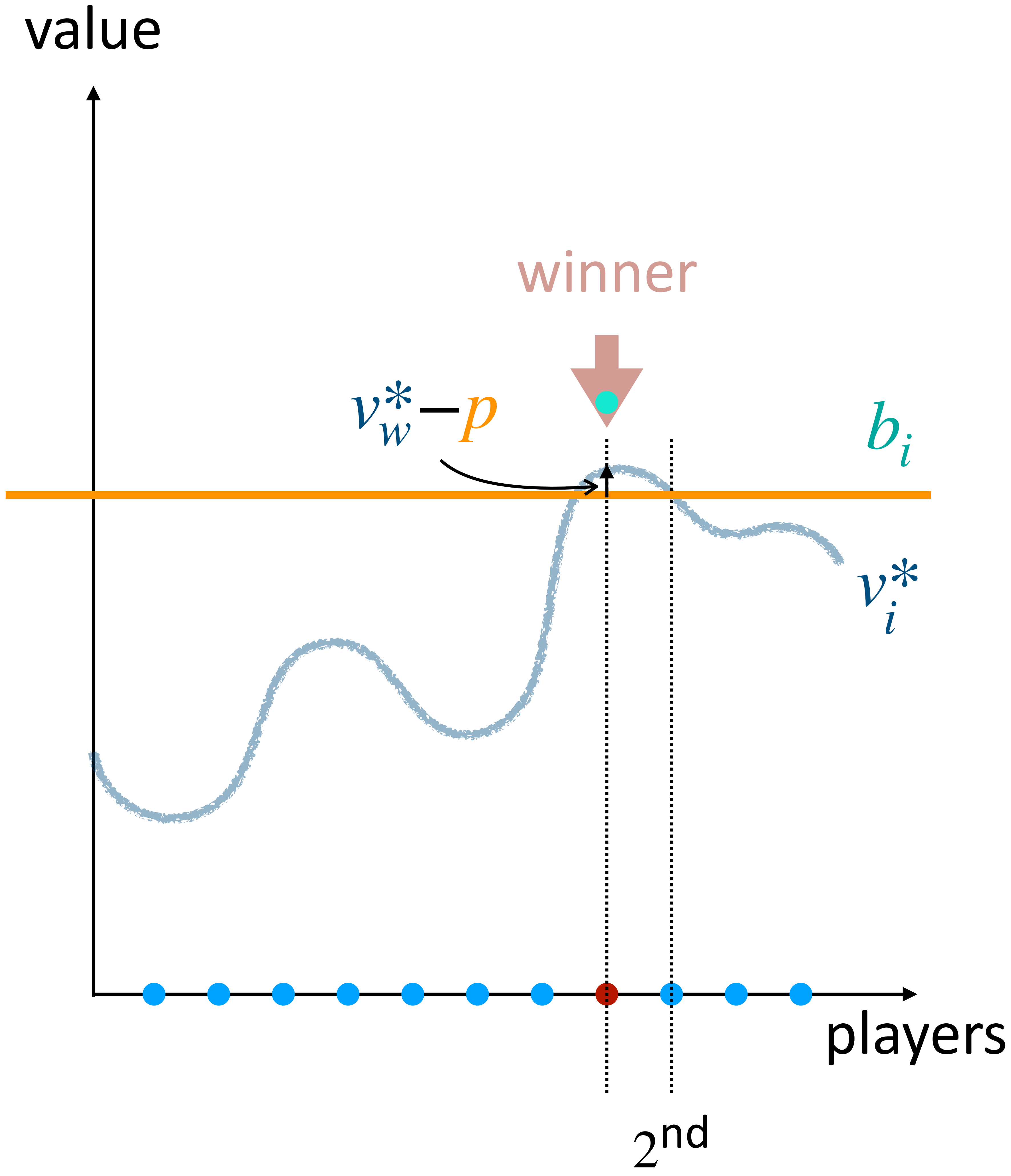

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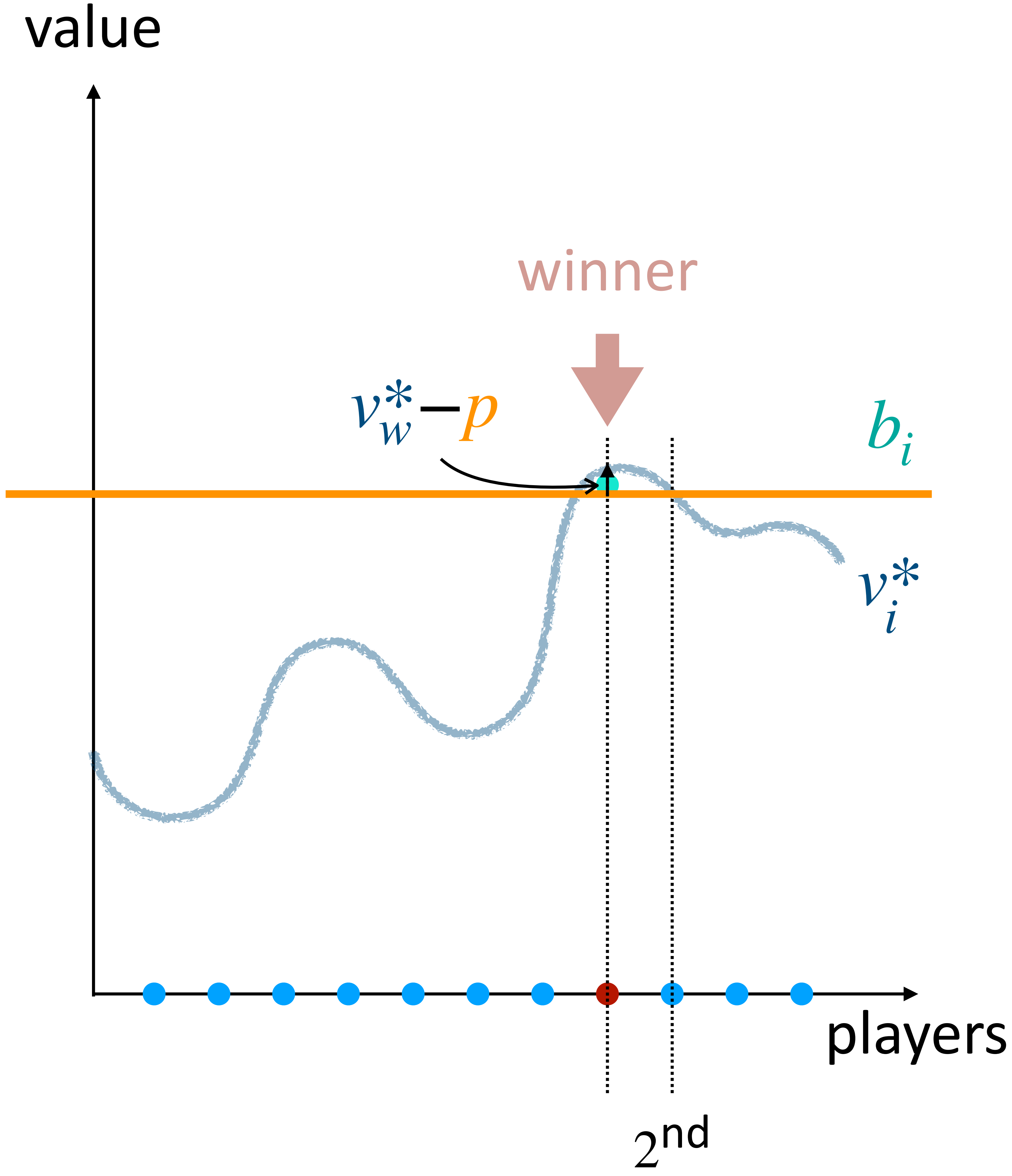

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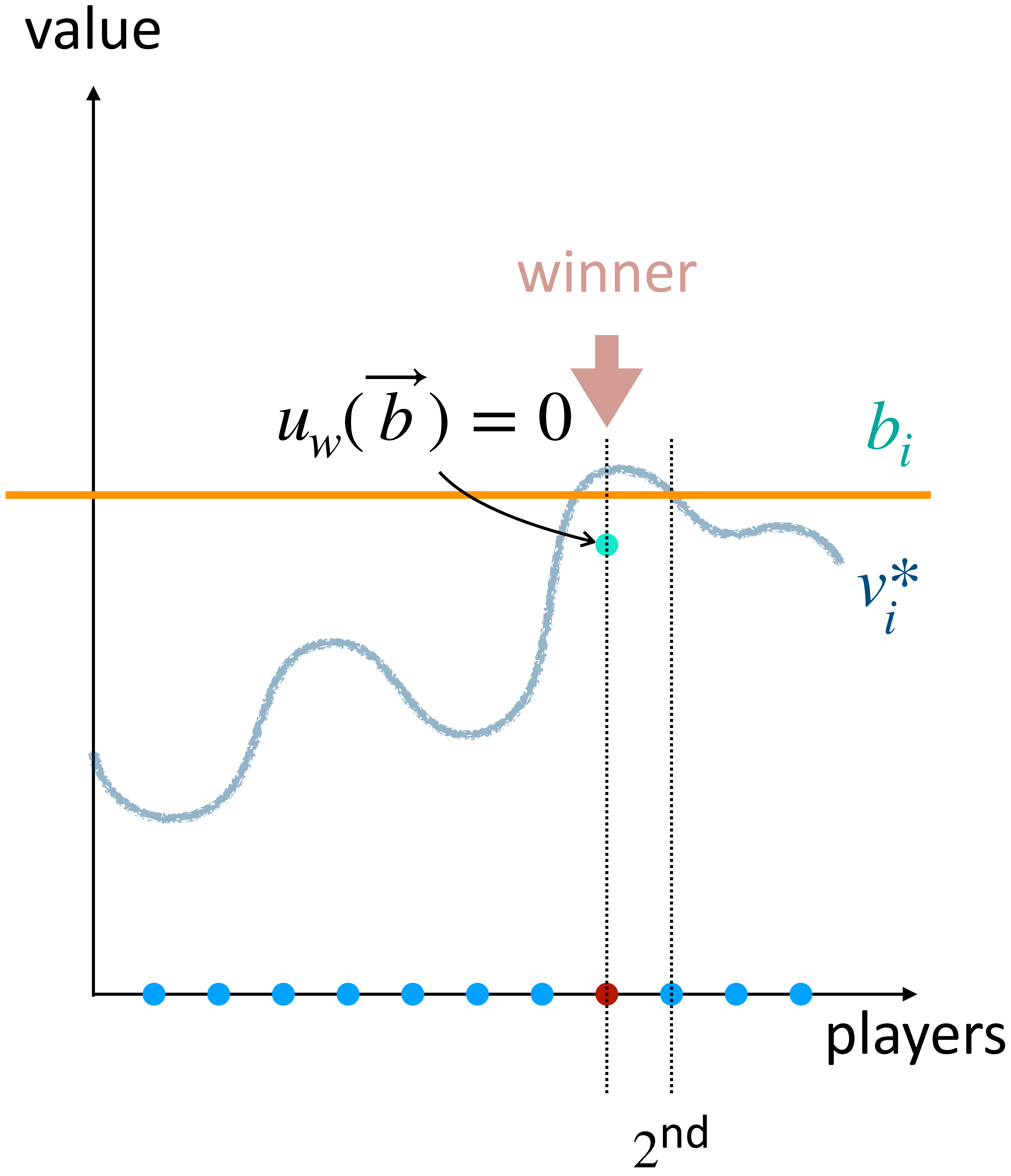

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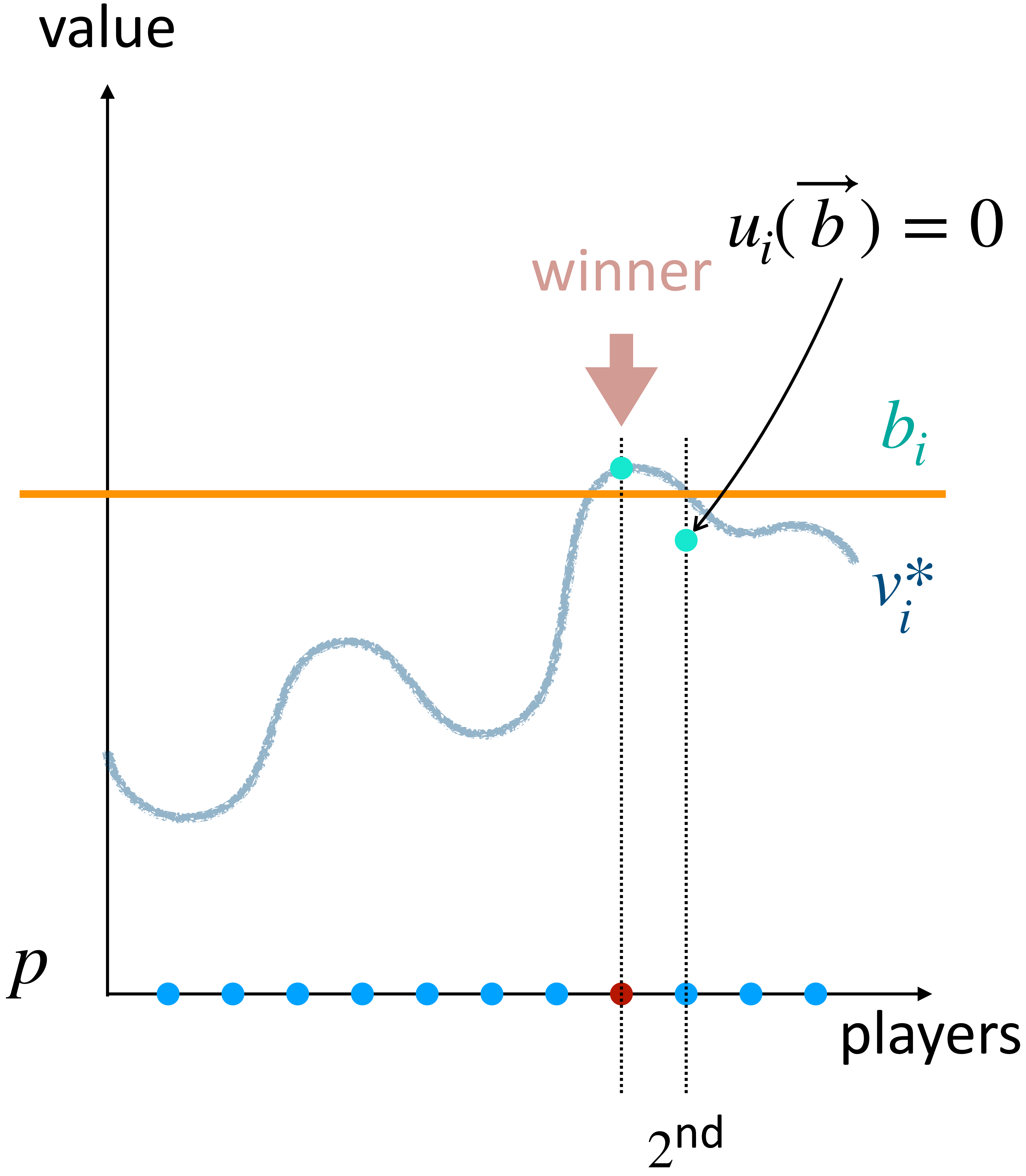

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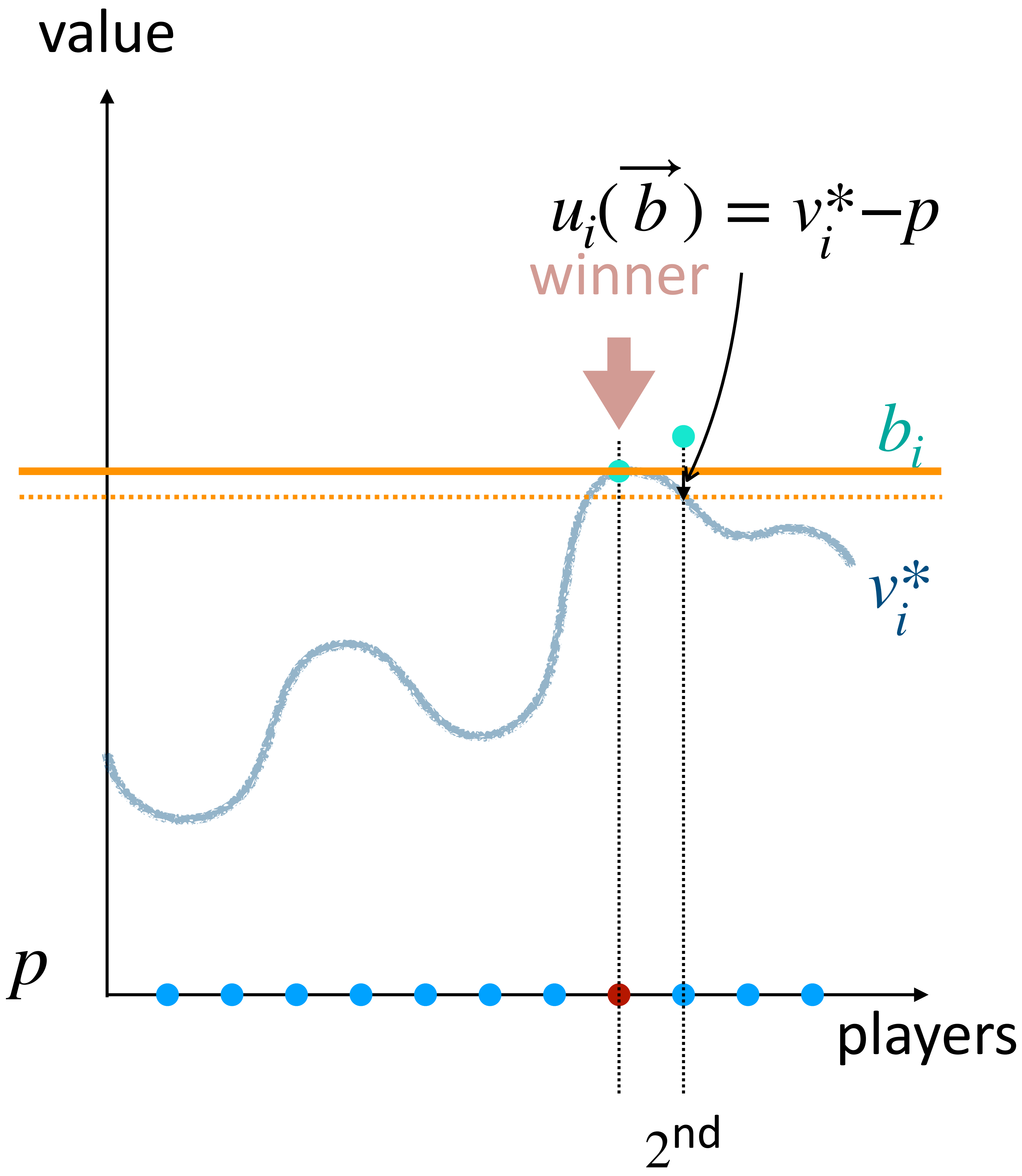

What happened

- In the auction game, Vickrey's second price auction (letting the winner pays the second-highest bid) is strategy-proof

Outline

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Mechanism Design

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  - The different members of society act *rationally* in a game theoretic sense
  - The preference of the participants are private
  - Examples: elections, markets, auctions, government policy, etc

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Vickrey-Clarke-Groves Mechanism

- A mechanism $(f, p_1, p_2, \cdots, p_n)$ is called a **Vickrey-Clarke-Groves (VCG) mechanism** if
  - $f(\vec{s}) \in \arg\max_{a \in A} \Sigma_i \, s_i(a)$, and
  - for some functions $h_1, h_2, \cdots, h_n$, where h_i is a function of $\overrightarrow{s_{-i}}$, we have that for all i: $p_i(\vec{s}) = h_i(\overrightarrow{s_{-i}}) - \Sigma_{j \neq i} \, s_j(f(\vec{s}))$

A: all possible outcome actions by the mechanism

$s_i(a)$: player i's (reported) utility/cost given the action a

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$s_i(a)$: player i's (reported) utility/cost given the action a

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VCG Mechanism is strategy-proof

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$$v_i^*(a) - p_i(a) = v_i^*(a) - h_i(\overrightarrow{s_{-i}}) + \Sigma_{j\neq i}\, v_j^*(a)$$

  - Since social welfare of $a^* = v_i^*(a^*) + \Sigma_{j\neq i} s_i(a^*) \geq v_i^*(a') + \Sigma_{j\neq i} s_i(a')$ for any a'

$$v_i^*(a^*) - p_i(a^*) \geq v_i^*(a) - p_i(a)$$

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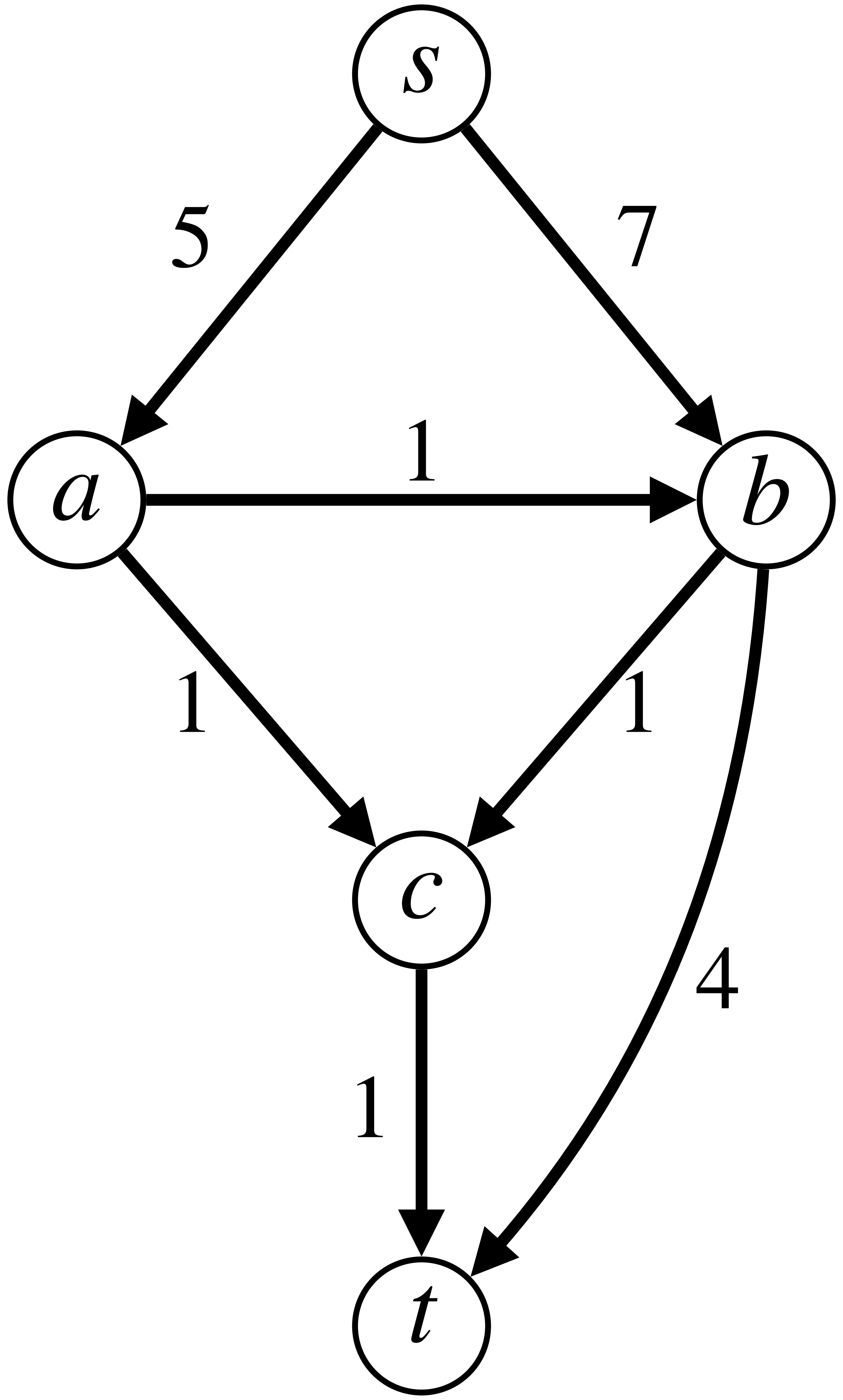
What happened

- The **Vickrey-Clarke-Groves (VCG) mechanism** is strategy-proof
  - As long as a mechanism is a VCG, it is strategy-proof

- **Clarke pivot rule**: $p_i(\vec{s}) = \max_{a \in A} \Sigma_{j \neq i} s_i(a) - \Sigma_{j \neq i}\, s_j(f(\vec{s}))$

Shortest Path

- Given a directed graph $G = (V, E)$, where each edge $e \in E$ is owned by a player. The player e has a (private) value v_e



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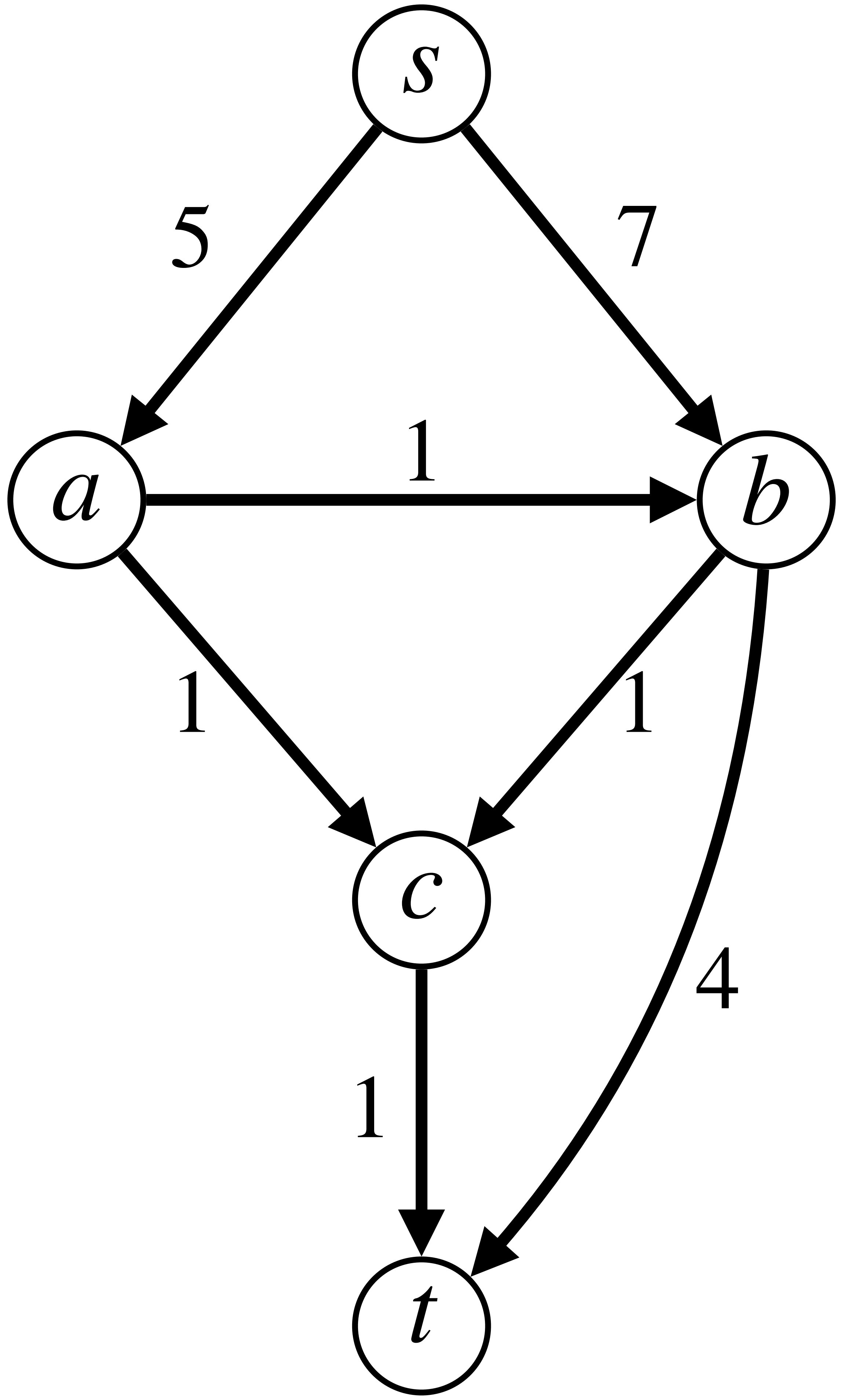

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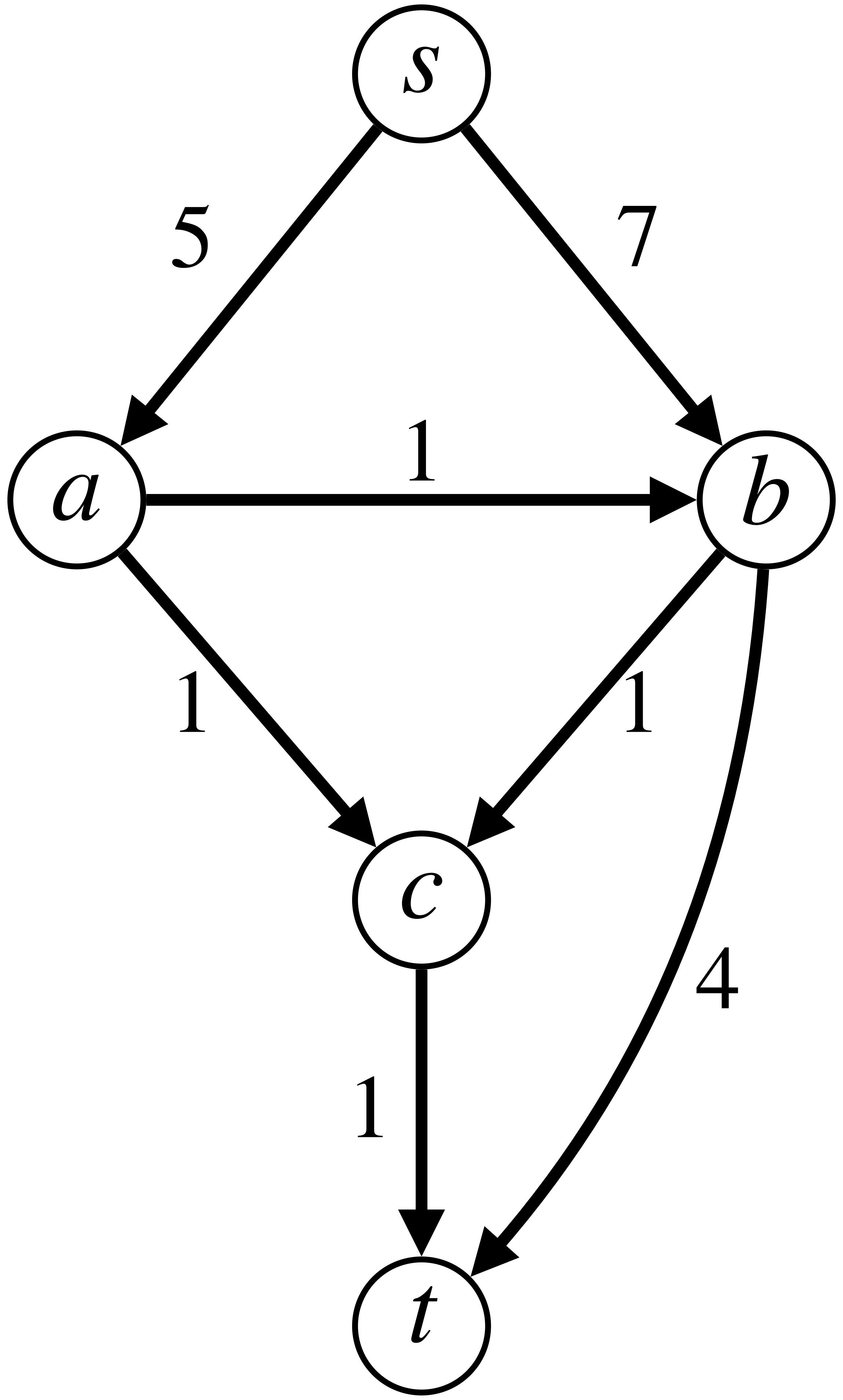
  - To minimize the social cost, which is Σ_e is expropriated $-v_e$, how should the government set a price for each player?



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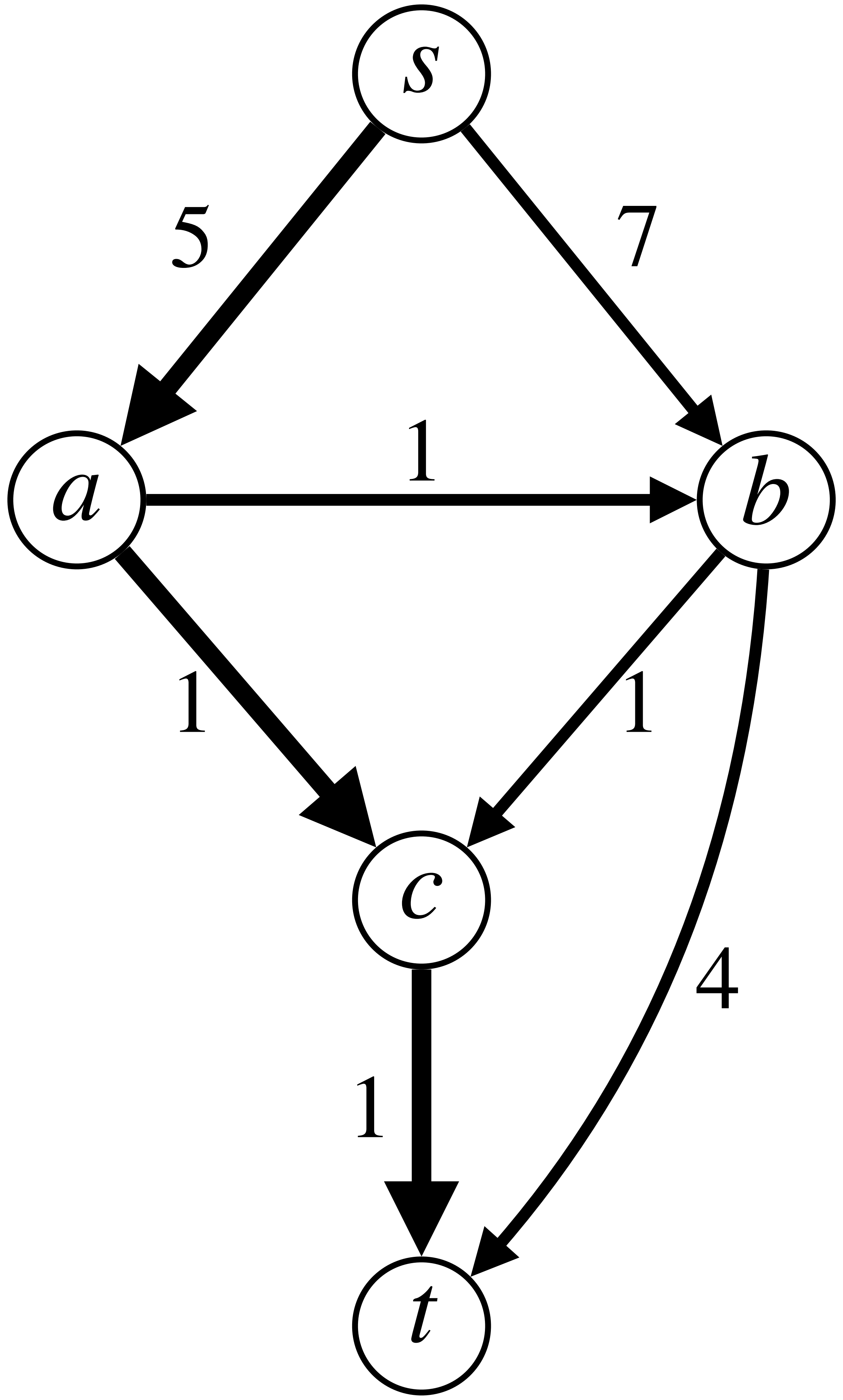

Shortest Path — Using VCG

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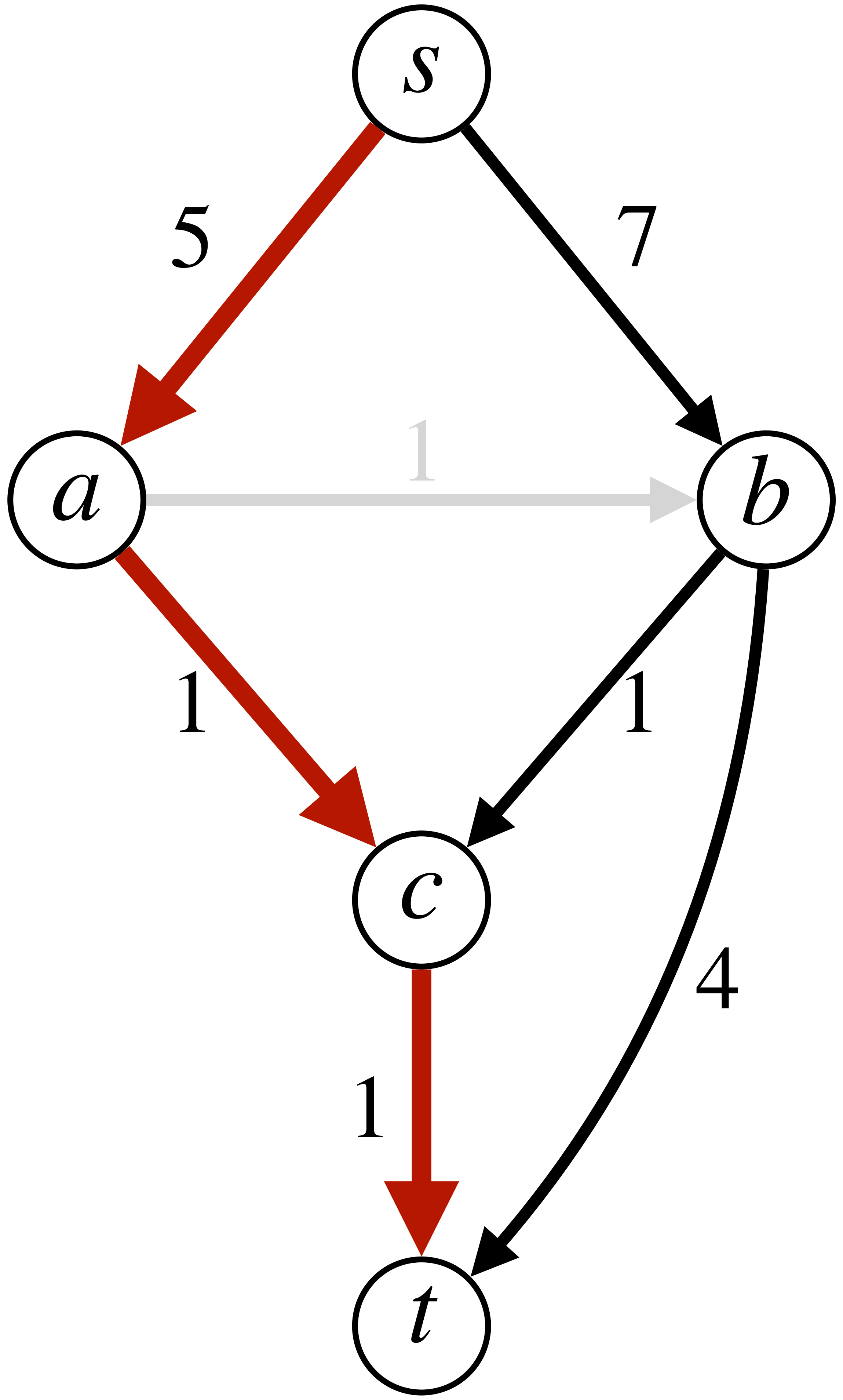

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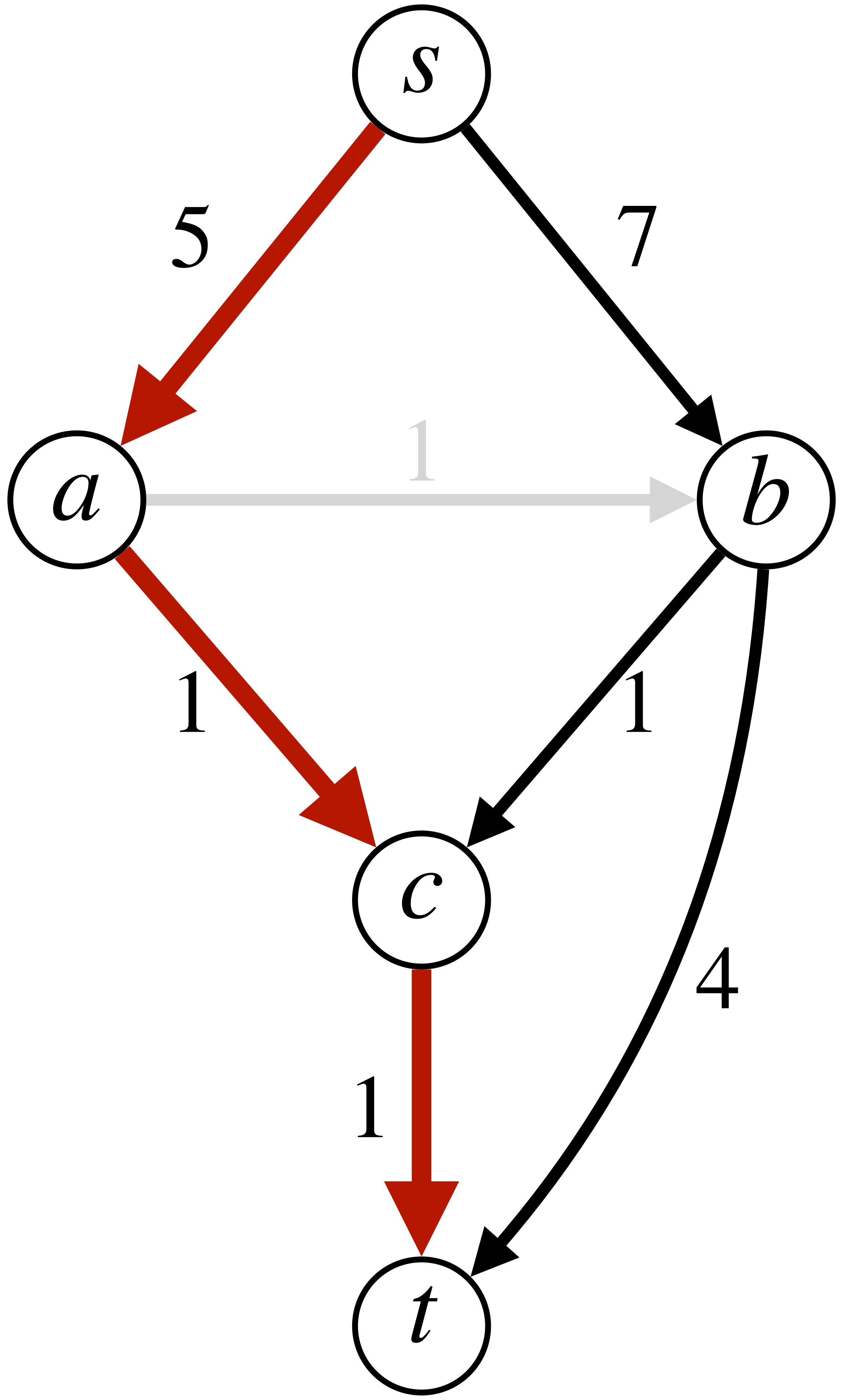

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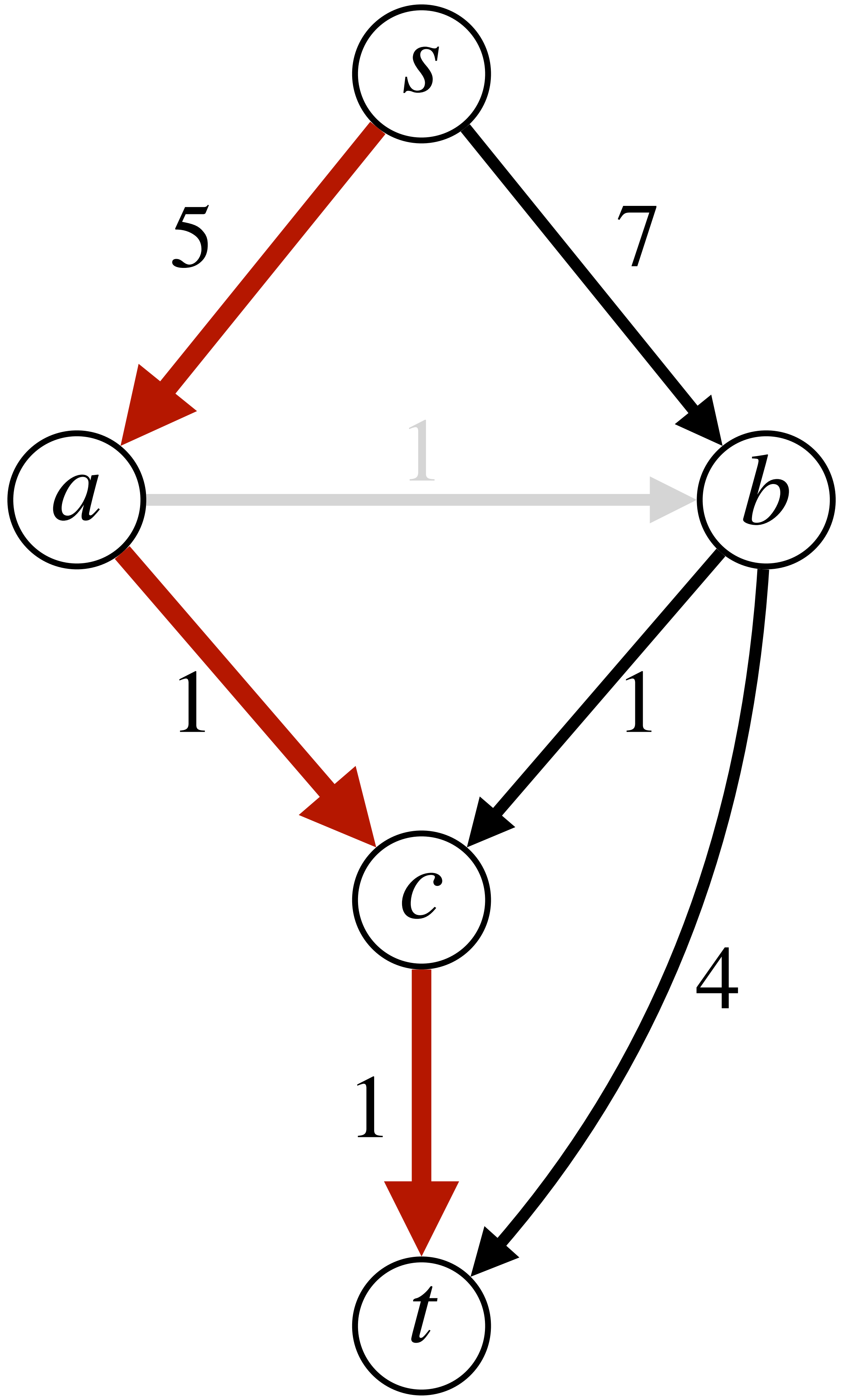

$$p_{(a,b)} = \max_{P' \text{ in } G'} \Sigma_{e \in P'}(-v_e) - \Sigma_{(a,b) \neq e \in P}(-v_e)$$
$$= -7 - (-7) \;= 0$$

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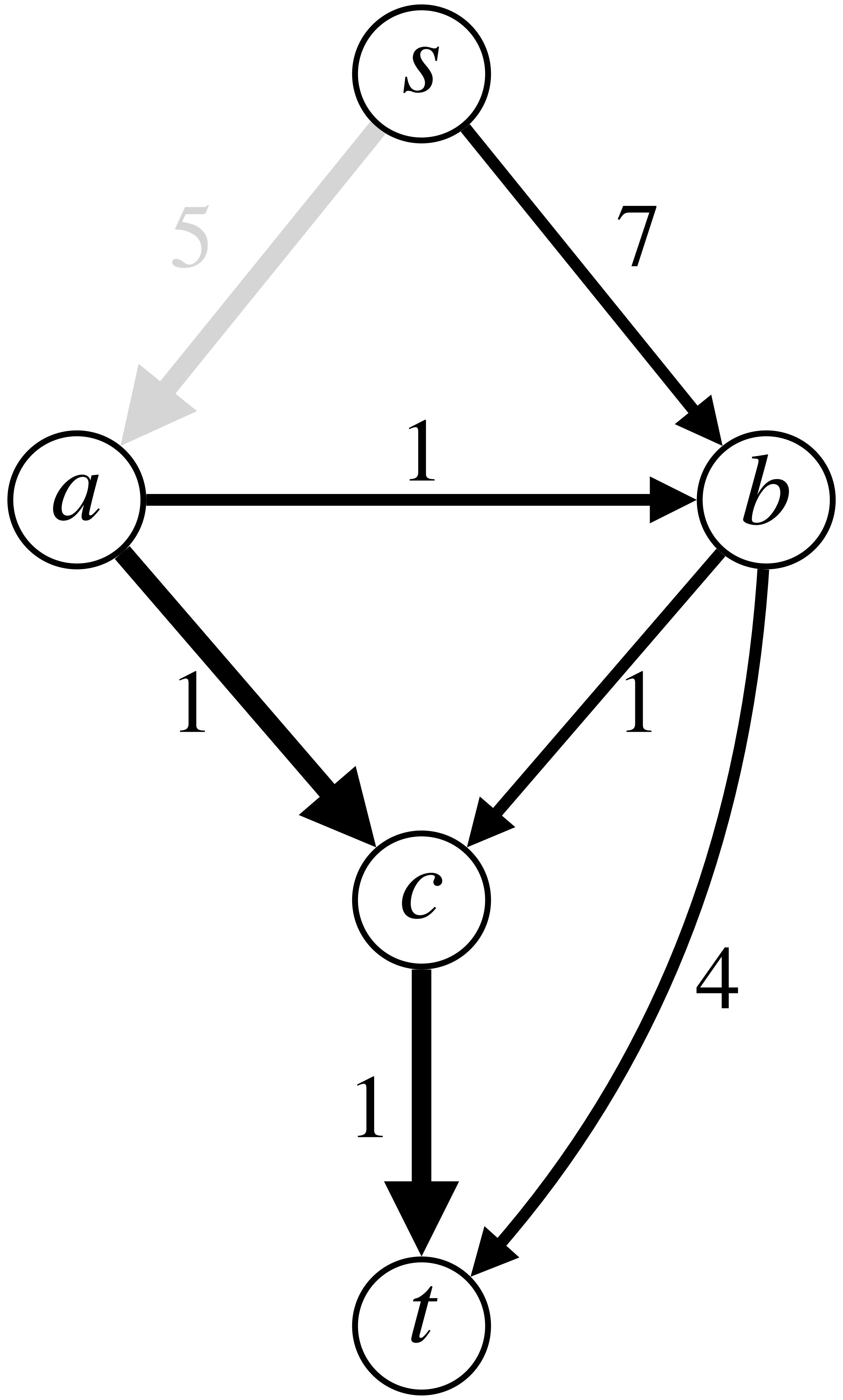

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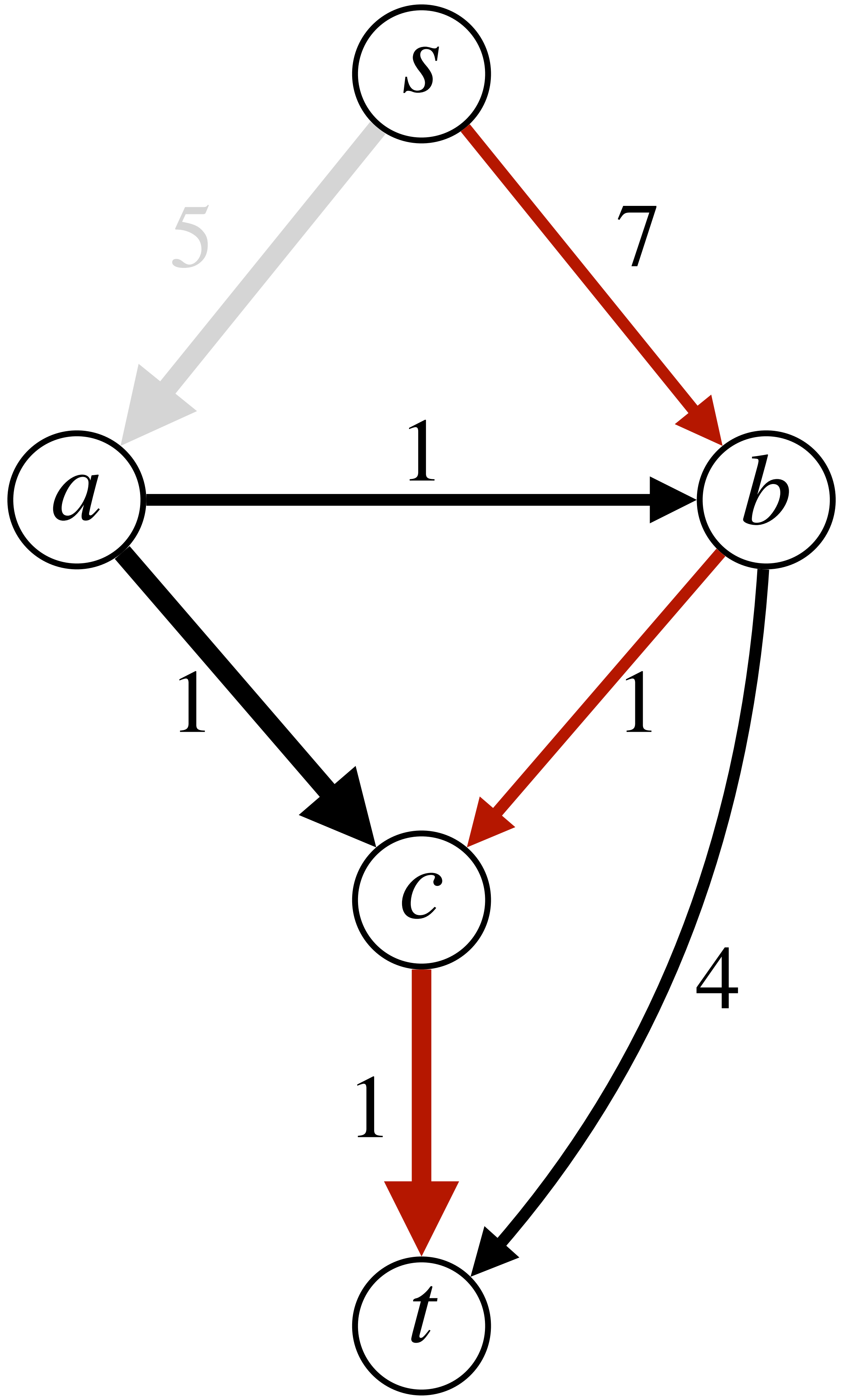

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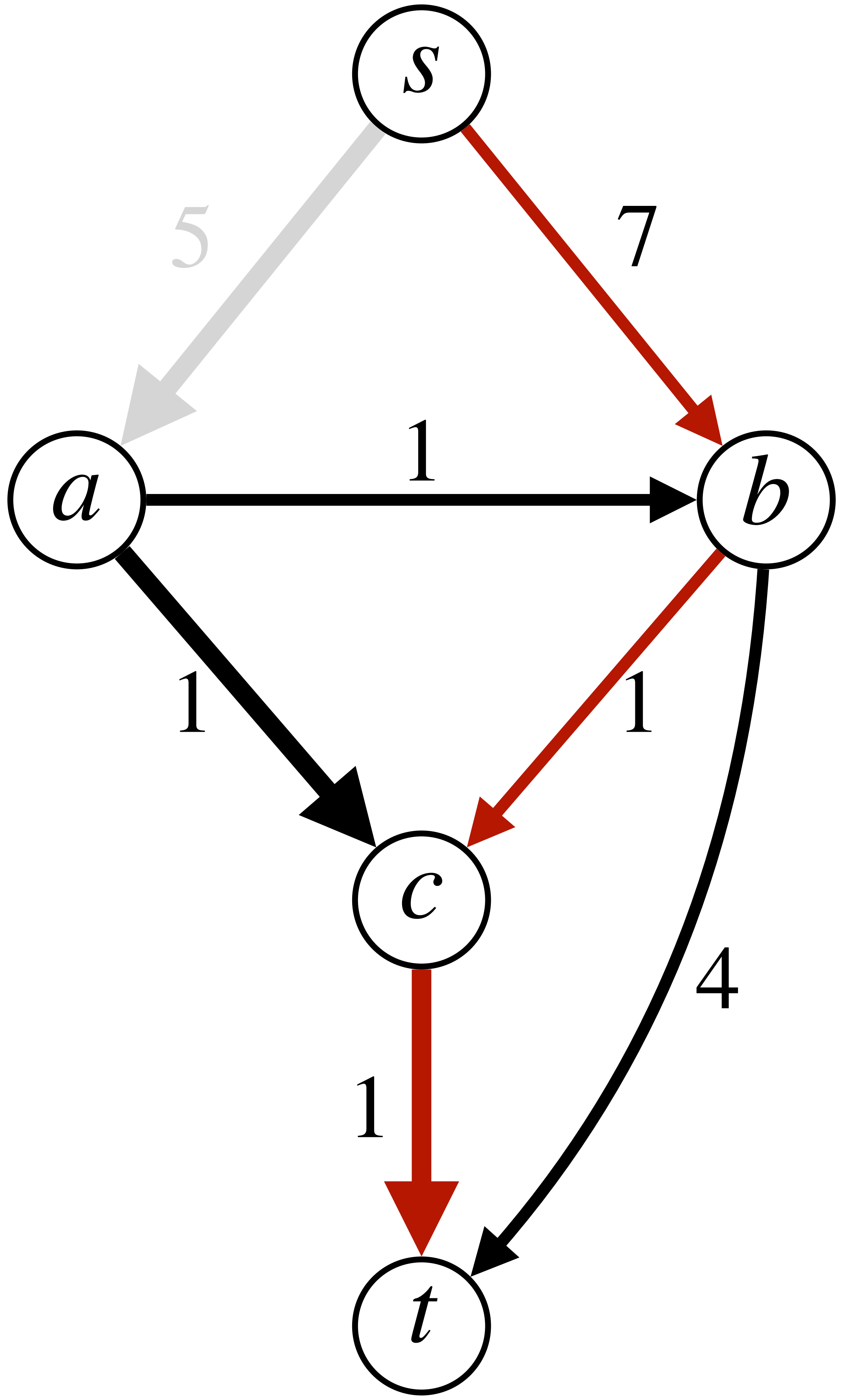

$$p_{(s,a)} = \max_{P' \text{ in } G'} \Sigma_{e \in P'}(-v_e) - \Sigma_{(s,a) \neq e \in P}(-v_e)$$
$$= -9 - (-2) \ = -7$$

Shortest Path — Using VCG

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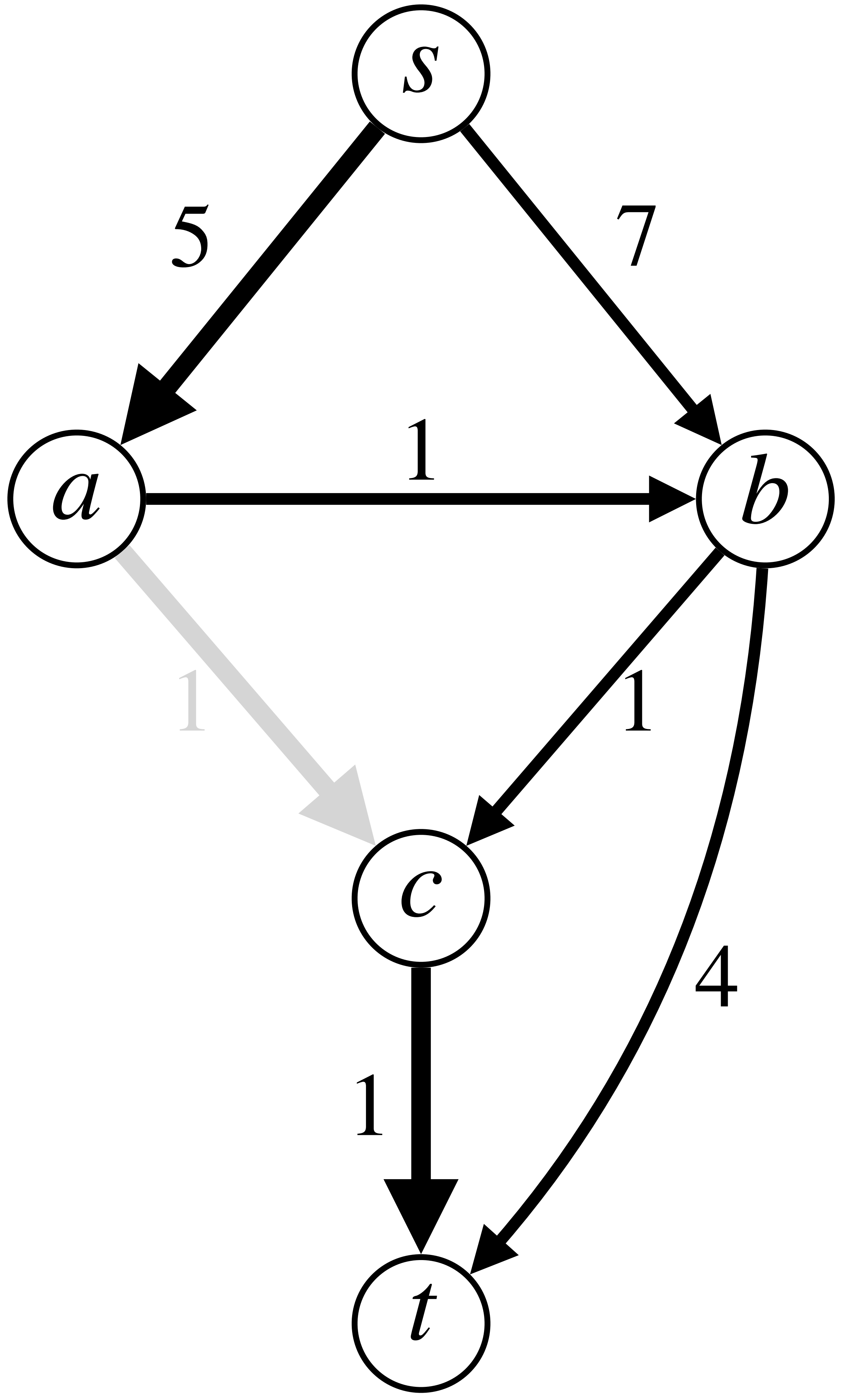

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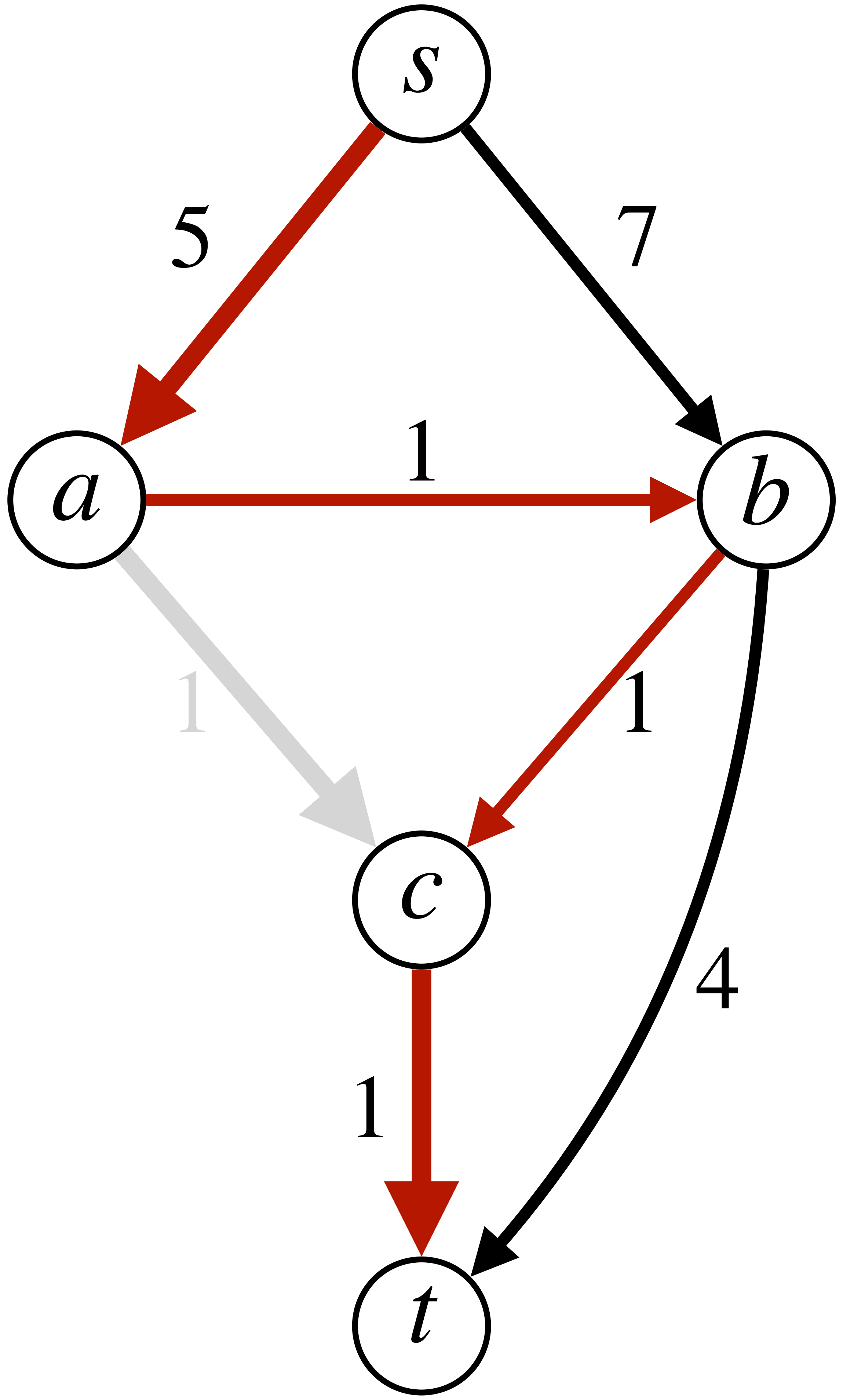

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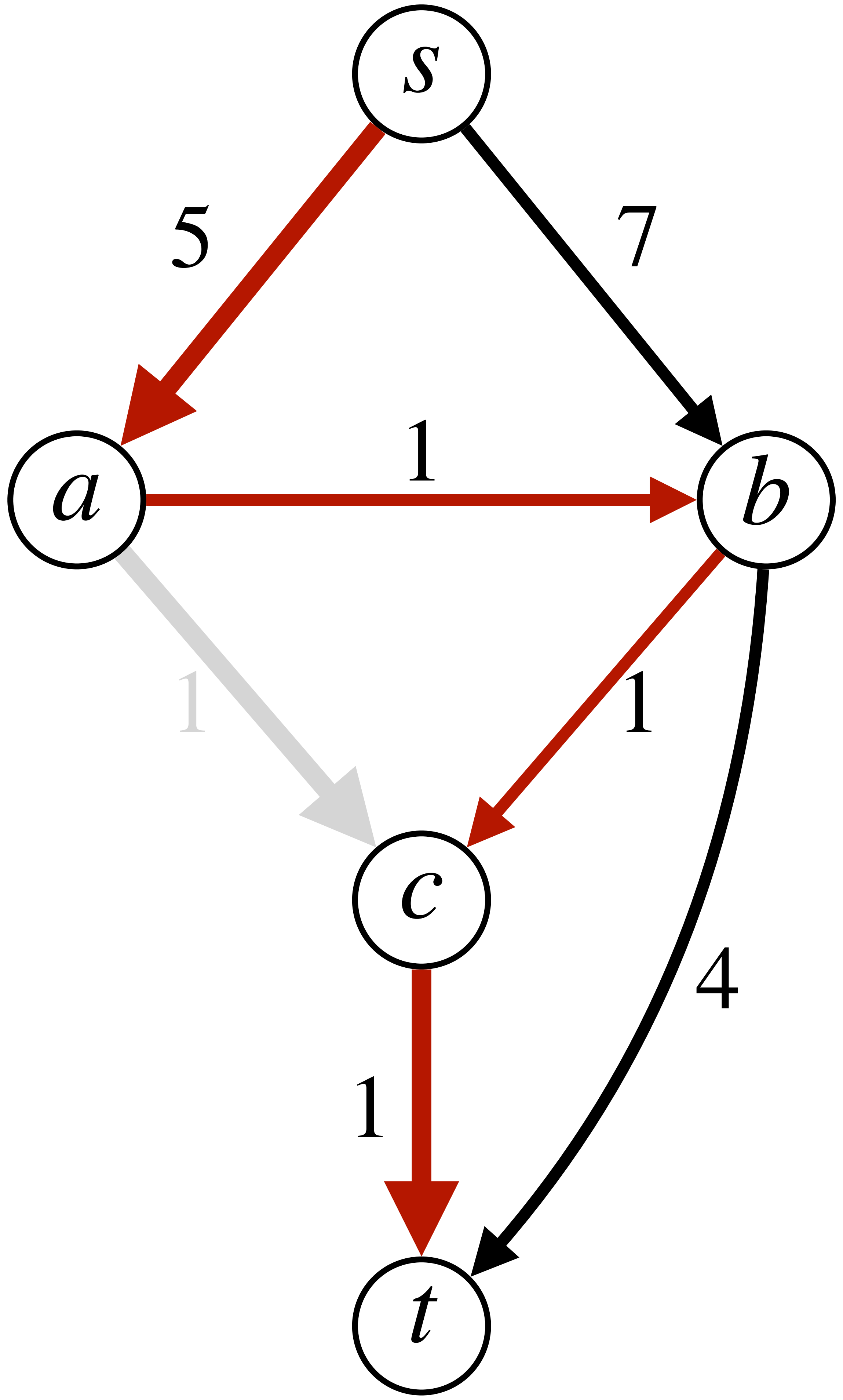

$$p_{(a,c)} = \max_{P' \text{ in } G'} \Sigma_{e \in P'}(-v_e) - \Sigma_{(a,c) \neq e \in P}(-v_e)$$
$$= -8 - (-6) \;\; = -2$$

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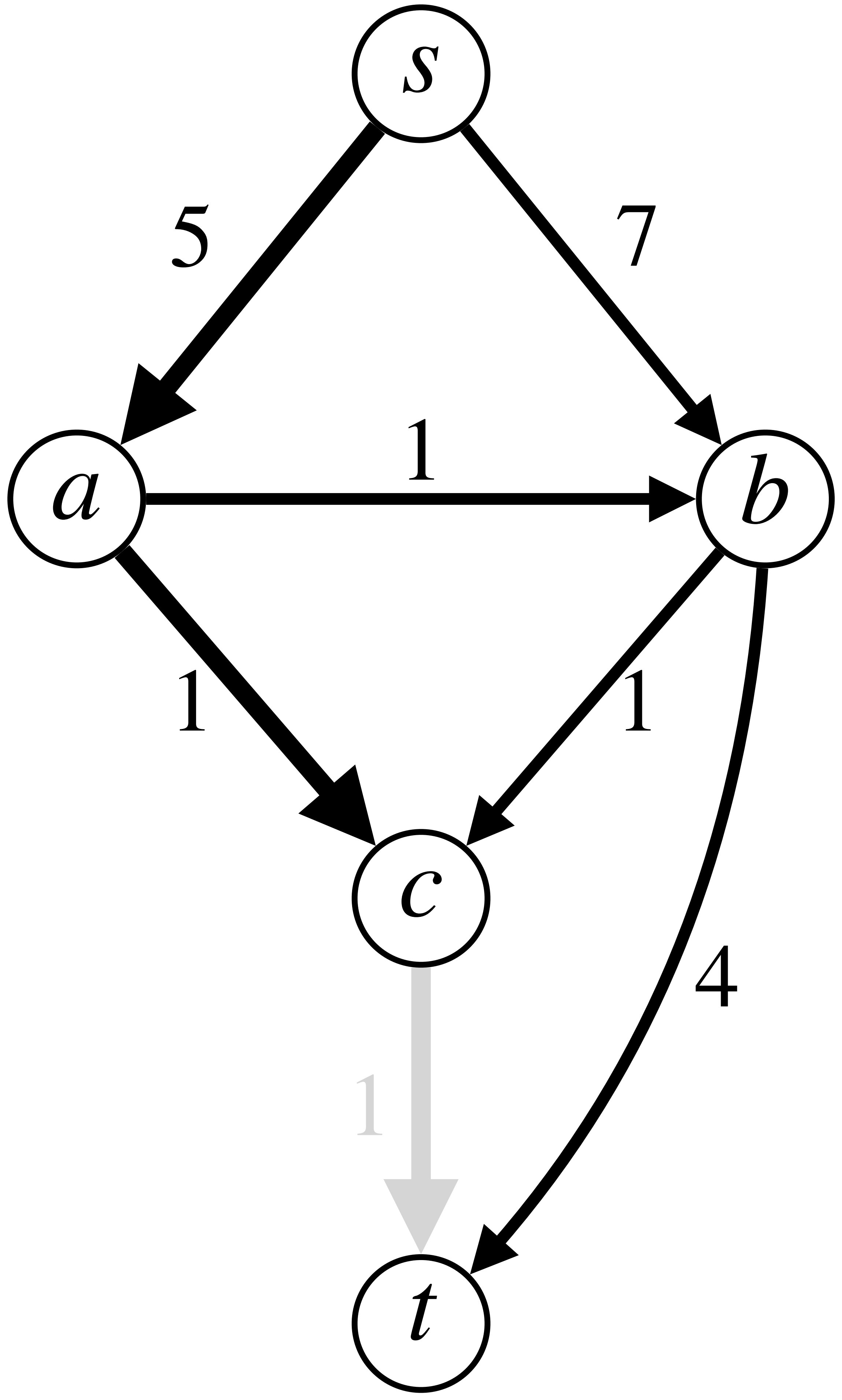

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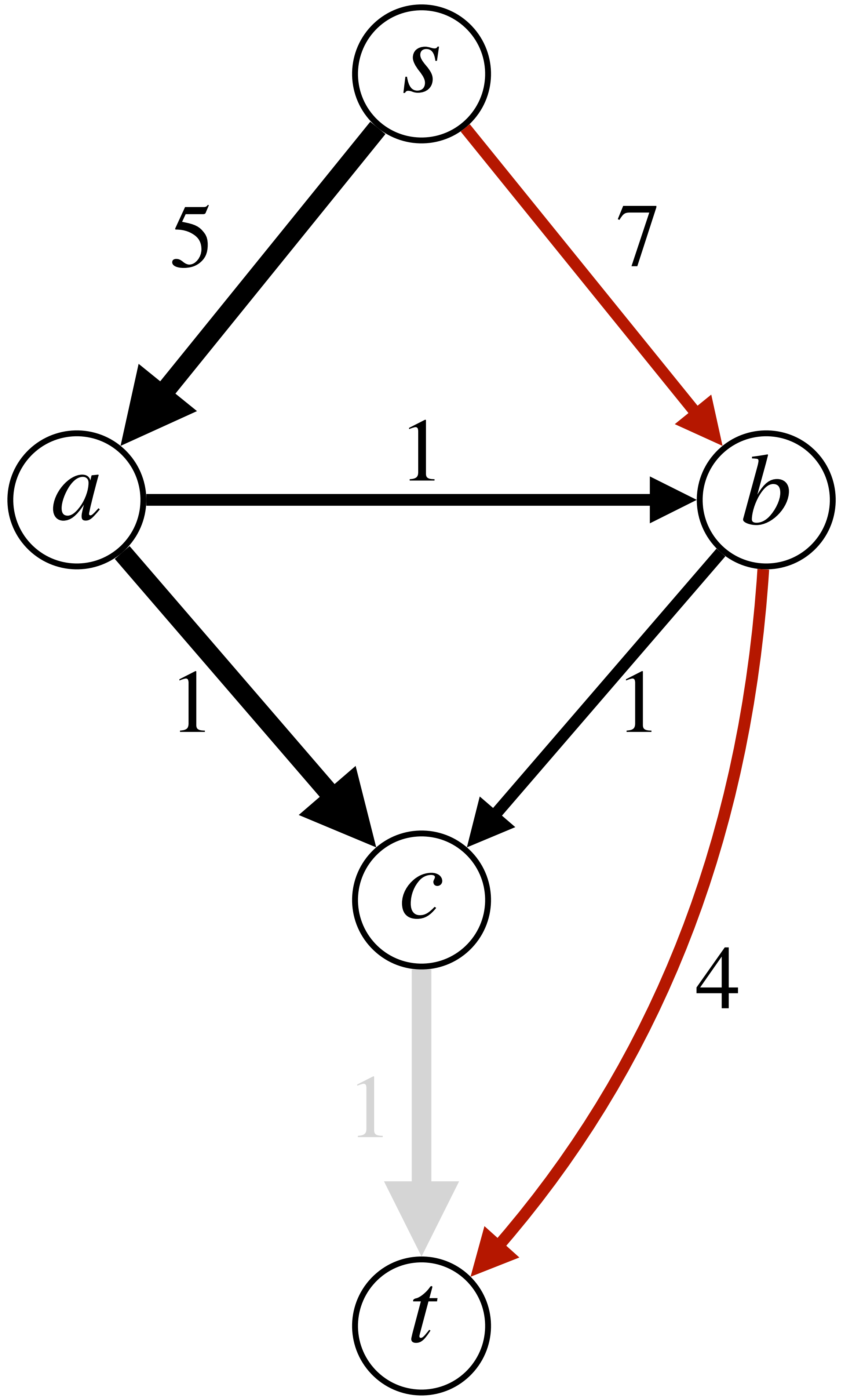

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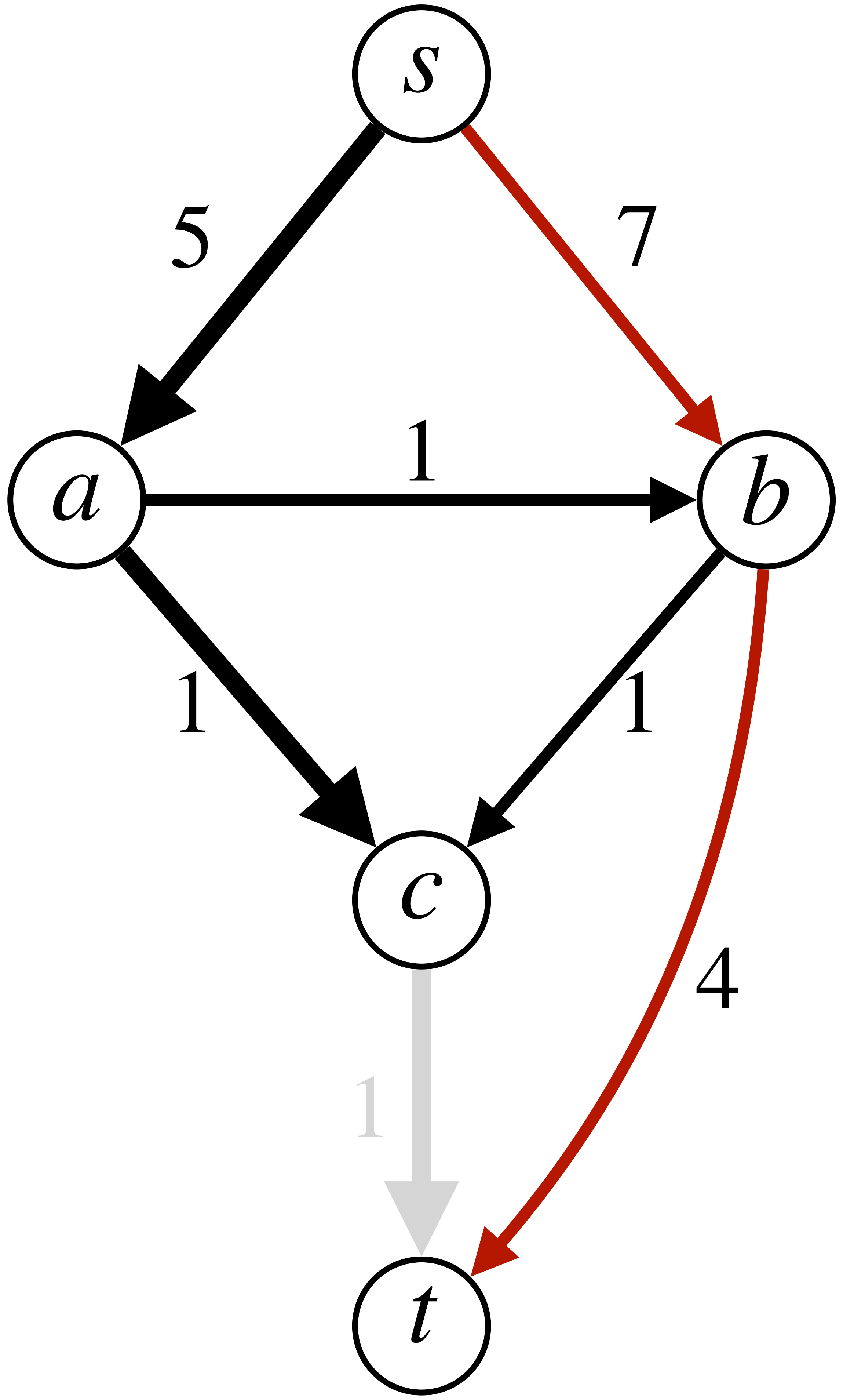

$$p_{(c,t)} = \max_{P' \text{ in } G'} \Sigma_{e \in P'}(-v_e) - \Sigma_{(c,t) \neq e \in P}(-v_e)$$
$$= -11 - (-6) = -5$$

Trade

- Seller has an item that costs v_s, and a potential buyer values it at v_b
  - If $v_b > v_s$, there is a grade. Otherwise ($v_b \leq v_s$), there is no trade

- How to make a price for the buyer (to pay) and a price for the seller (to receive) so the buyers and sellers report their value truthfully?

Trade — Using VCG mechanism

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Clarke pivot rule: $p_i(\vec{s}) = \max\limits_{a \in A} \Sigma_{j \neq i} s_i(a) - \Sigma_{j \neq i} s_j(f(\vec{s}))$

- if $v_b > v_s$: $p_s(v_s, v_b) = \max\limits_{d \in \{0,1\}} v_b(d) - v_b = v_b - v_b = 0$

- if $v_b \leq v_s$: $p_s(v_s, v_b) = \max\limits_{d \in \{0,1\}} v_b(d) - 0 = v_b - 0 = v_b$

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  - If $v_b > v_s$, there is a grade. Otherwise ($v_b \leq v_s$), there is no trade

Clarke pivot rule: $p_i(\vec{s}) = \max_{a \in A} \Sigma_{j \neq i} s_i(a) - \Sigma_{j \neq i} s_j(f(\vec{s}))$

- if $v_b > v_s$: $p_s(v_s, v_b) = \max_{d \in \{0,1\}} v_b(d) - v_b = v_b - v_b = 0$

$$p_b(v_s, v_b) = \max_{d \in \{0,1\}} - v_s(d) - (-v_s) = 0 + v_s = v_s$$

- if $v_b \leq v_s$: $p_s(v_s, v_b) = \max_{d \in \{0,1\}} v_b(d) - 0 = v_b - 0 = v_b$

$$p_b(v_s, v_b) = \max_{d \in \{0,1\}} - v_s(d) - 0 = 0 - 0 = 0$$