Exercise 14: Algorithmic Game Theory

1. Consider the selfish load balancing game on $m = 3$ machines. Let the processing times of the players be $\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5\}$. Find a Nash equilibrium that is as inefficient as possible.

   The optimal solution is to put the two jobs with processing times 5 on one machine and evenly distribute the rest of the jobs on the remaining two machines (that is, each machine has 4 jobs with processing times $1, 2, 3$, and 4, respectively). In this case, the optimal cost is 10.

   Consider the following assignment:

   - On machine 1, there are three jobs $3, 4, 5$,
   - On machine 2, there are three jobs $2, 3, 4$, and
   - On machine 3, there are four jobs $1, 1, 2, 5$.

   This assignment is a Nash equilibrium, and its social cost is 12 (machine 1). That is, its PoA is 1.2.

2. In the lecture, we showed that the Vickrey-second price mechanism is strategy-proof. Use the VCG mechanism to deduct the Vickrey-second price mechanism.

   Let a be the winner of the strategy vector $\vec{b}$ and a' be the winner of the strategy vector $\vec{b_{-i}}$. By VCG mechanism and picking Clarke pivot rule, the price of player i is

   $$\sum_{j \neq i} b_j(a') - \sum_{j \neq i} b_j(a),$$

   where $b_x(w) = b_x$ if $x = w$ and $b_x(w) = 0$ otherwise.

   If player i is the winner in the original game (that is, $a = i$), $p_i = \sum_{j \neq i} b_j(a') - \sum_{j \neq i} b_j(a) = \max_{j \neq i} b_j - 0$. That is, p_i is the second-highest bid.

   If player i does not win in the original game (that is, $a \neq i$), $p_i = \sum_{j \neq i} b_j(a') - \sum_{j \neq i} b_j(a) = \max_{j \neq i} b_j - \max_{j \neq} b_j$ =0.

   Hence, the Veckrey-second price mechanism is exactly an application of the VCG mechanism with the Clarke pivot rule.

3. In the shortest path game, what happens if a player misreported its value to its edge? Concentrate on edge e. Let P denote the shortest path from s to t in the original graph, and P' be the shortest path after removing edge e. Let v_e^*, w_e, $\ell(P)$, and $\ell(P')$ be the true length of e, claimed length of e, P and P', respectively.

   First, consider the case where $e \in P$. The utility under truthfully reporting v_e^* is $u_e^* = v_e^* - p_e = v_e^* - \ell(P) + \ell(P') = v_e^*$. If the player e reports $w_e > v_e^*$ such that $e \notin P'$, $p_e = \ell(P) - w_e - \ell(P')$. The utility under claiming w_e is $v_e^* - p_e = v_e^* - \ell(P) + \ell(P') + w_e < -\ell(P) + \ell(P') < u_e^*$.

   On the other hand, consider the case where $e \notin P$. The utility under truthfully reporting v_e^* is $u_e^* = -\ell(P) + \ell(P') = 0$.

   If the player reports $w_e < v_e^*$ such that $e \in P'$, $p_e = \ell(P) - (\ell(P') - w_e) = \ell(P) - \ell(P') + w_e$. The utility under claiming w_e is $v_e^* - p_e = v_e^* - \ell(P) + \ell(P') - w_e < 0$ since $\ell(P) \leq \ell(P') - (v_e^* - w_e)$.