Randomized (Online) Algorithms

Part 2

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Algorithms for Decision Support 2024

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Lower bounds

How to find a lower bound for all randomized algorithms for a problem?

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- Show that for a fixed probability distribution over the input, all deterministic algorithms perform "bad" "on average"?

Lower bounds

How to find a lower bound for all randomized algorithms for a problem?

- Show that for every deterministic algorithm we can find a "bad" input?
  - **No!** There still might be a "good" randomized algorithm that chooses between them.
- Show that for every possible randomized algorithm, there exists a hard instance?
  - **Yes**... but can be challenging.
- Show that for a fixed probability distribution over the input, all deterministic algorithms perform "bad" "on average"?
  - **Yao's Principle!**

Yao's Principle

Yao's Principle, informal

The worst-case performance of the best randomized algorithm is equal to the average performance of the best deterministic algorithm on the worst distribution of the inputs

- Yao suggests that we look at the worst distribution on the inputs that we can think of.
- And claim that every deterministic algorithm does not have, on average, a performance better than c on this set of inputs.
- Then, by Yao's Principle, this is a lower bound on the worst-case performance of randomized algorithms.

Yao's Principle

Assumptions.

- We have a **finite** class of **deterministic** algorithms $\mathcal{A}$
- We have a **finite** class of instances $\mathcal{I}$.

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- Let $\mathbf{Pr}_{\mathrm{RAND}}$ be a probability distribution over $\mathcal{A}$.

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Algorithm $\textsc{Alg} \in \mathcal{A}$ on the class of instances $\mathcal{I}$ has expected cost $\mathbb{E}_{\text{ADV}}\left[\textsc{Alg}(\mathcal{I})\right] = \sum_{I \in \mathcal{I}} \mathbf{Pr}_{\text{ADV}}(I) \cdot \textsc{Alg}(I)$.

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$\mathbb{E}_{\text{ADV}}\left[\textsc{Alg}(\mathcal{I})\right] = \sum_{I \in \mathcal{I}} \mathbf{Pr}_{\text{ADV}}(I) \cdot \textsc{Alg}(I)$.

Algorithm $\textsc{Rand}$ on instance $I \in \mathcal{I}$ has expected cost
$\mathbb{E}_{\text{RAND}}\left[\textsc{Rand}(I)\right] = \sum_{\textsc{Alg} \in \mathcal{A}} \mathbf{Pr}_{\text{RAND}}(\textsc{Alg}) \cdot \textsc{Alg}(I)$.

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Algorithm RAND on instance $I \in \mathcal{I}$ has expected cost
$\mathbb{E}_{\text{RAND}}\left[\text{RAND}(I)\right] = \sum_{\text{ALG} \in \mathcal{A}} \mathbf{Pr}_{\text{RAND}}(\text{ALG}) \cdot \text{ALG}(I)$.

Algorithm RAND on the class of instances $\mathcal{I}$ has expected cost
$\mathbb{E}_{\text{ADV}}\left[\mathbb{E}_{\text{RAND}}\left[\text{RAND}(\mathcal{I})\right]\right] = \sum_{\text{I} \in \mathcal{I}} \mathbf{Pr}_{\text{ADV}}(\text{I}) \cdot \mathbb{E}_{\text{RAND}}\left[\text{RAND}(I)\right]$.

Yao's Principle

Yao's Principle

If,

$$\frac{\min_{\textsc{Alg} \in \mathcal{A}} \mathbb{E}_{\textsc{Adv}}\left[\textsc{Alg}(\mathcal{I})\right]}{\mathbb{E}_{\textsc{Adv}}\left[\textsc{Opt}(\mathcal{I})\right]} \geq c$$

Then, for every randomized algorithm $\textsc{Rand}$, there exists an instance $I \in \mathcal{I}$, such that,

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(I)\right]}{\textsc{Opt}(I)} \geq c$$

Yao's Principle

Proof.

We first rewrite the expected cost of $\textsc{Rand}$ with respect to $\mathbf{Pr}_{\textsc{Adv}}$.

$$\begin{aligned}
&\mathbb{E}_{\textsc{Adv}}\left[\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(\mathcal{I})\right]\right] \\
&= \sum_{\mathrm{I}\in\mathcal{I}} \mathbf{Pr}_{\textsc{Adv}}\left(\mathrm{I}\right) \cdot \mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(I)\right]
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Now assume that

$$\frac{\min_{\textsc{Alg} \in \mathcal{A}} \mathbb{E}_{\textsc{Adv}}\left[\textsc{Alg}(\mathcal{I})\right]}{\mathbb{E}_{\textsc{Adv}}\left[\textsc{Opt}(\mathcal{I})\right]} \geq c$$

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&\geq \sum_{\textsc{Alg} \in \mathcal{A}} \mathbf{Pr}_{\textsc{Rand}} (\textsc{Alg}) \cdot c \cdot \mathbb{E}_{\textsc{Adv}} [\textsc{Opt}(\mathcal{I})] \\
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$$\mathbb{E}_{\text{ADV}}\left[\mathbb{E}_{\text{RAND}}\left[\text{RAND}(\mathcal{I})\right]\right] \geq c \cdot \mathbb{E}_{\text{ADV}}\left[\text{OPT}(\mathcal{I})\right]$$

Claim: There exists an instance $I \in \mathcal{I}$ such that,

$$\mathbb{E}_{\text{RAND}}\left[\text{RAND}(I)\right] \geq c \cdot \text{OPT}(I)$$

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Claim: There exists an instance $I \in \mathcal{I}$ such that,

$$\mathbb{E}_{\text{RAND}}\left[\text{RAND}(I)\right] \geq c \cdot \text{OPT}(I)$$

Suppose aiming towards a contradiction that for every $I \in \mathcal{I}$

$$\mathbb{E}_{\text{RAND}}\left[\text{RAND}(I)\right] < c \cdot \text{OPT}(\mathrm{I})$$

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We now have,

$$\mathbb{E}_{\textsc{Adv}}\left[\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(\mathcal{I})\right]\right] \geq c \cdot \mathbb{E}_{\textsc{Adv}}\left[\textsc{Opt}(\mathcal{I})\right]$$

Claim: There exists an instance $I \in \mathcal{I}$ such that,

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Suppose aiming towards a contradiction that for every $I \in \mathcal{I}$

$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(I)\right] < c \cdot \textsc{Opt}(\mathrm{I})$$

It follows that $$\sum_{I\in\mathcal{I}} \mathbf{Pr}_{\textsc{Adv}}(I) \cdot \mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(I)\right] < c \cdot \sum_{I\in\mathcal{I}} \mathbf{Pr}_{\textsc{Adv}}(I) \cdot \textsc{Opt}(I)$$

Contradiction! □

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Assuming that

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There exists an instance $I \in \mathcal{I}$ such that,

$$\mathbb{E}_{\text{RAND}}\left[\text{RAND}(I)\right] \geq c \cdot \text{OPT}(I)$$

This proves Yao's Principle $\square$

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Paging

Last time we have seen that there exists a randomized algorithm for the online paging problem that is $2H_k$-competitive in expectation.

Is it possible to do better?

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Last time we have seen that there exists a randomized algorithm for the online paging problem that is $2H_k$-competitive in expectation.

Is it possible to do better?

- Can we find a matching lower bound? **Yao's Principle!**

Harmonic number

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$$

Coupon Collector

Harmonic number

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$$

Coupon Collector's Problem

If each box of breakfast cereal contains a coupon, and there are n different coupons, how many boxes do you need to buy in expectation to collect all n coupons?

Coupon Collector

Harmonic number

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$$

Coupon Collector's Problem

If each box of breakfast cereal contains a coupon, and there are n different coupons, how many boxes do you need to buy in expectation to collect all n coupons?

Theorem

For the Coupon Collector's problem, the expected value of purchases required in order to collect each of the n coupons at least once is nH_n.

Coupon Collector

Proof.
Let X be the discrete random variable that represents the number of purchases until each of the n coupons is collected.

Coupon Collector

Proof.
Let X be the discrete random variable that represents the number of purchases until each of the n coupons is collected.
Le X_i be the discrete random variable that represents the number of purchases, after the $(i-1)$-th distinct coupon is collected, to collect the i-th distinct coupon.

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Let X be the discrete random variable that represents the number of purchases until each of the n coupons is collected.
Le X_i be the discrete random variable that represents the number of purchases, after the $(i-1)$-th distinct coupon is collected, to collect the i-th distinct coupon.
Then,

$$X = \sum_{i=1}^{n} X_i$$

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Then,

$$X = \sum_{i=1}^{n} X_i$$

Let $\mathcal{E}_i$ be the event that one of those coupons is collected in the next purchase.
Then,

$$\mathbf{Pr}\left(\mathcal{E}_i\right) = \frac{n-(i-1)}{n}$$

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Then,

After the $(i-1)$-th distinct coupon is collected, there are $n-i+1$ coupons remaining to be collected.

Let $\mathcal{E}_i$ be the event that one of those co[…]s is collected in the next purchase.

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$$\mathbf{Pr}\left(\mathcal{E}_i\right) = \frac{n-(i-1)}{n}$$

X_i follows a **geometric distribution**.

$$\mathbb{E}_{\text{Adv}}\left[X_i\right] = \frac{1}{\mathbf{Pr}\left(\mathcal{E}_i\right)} = \frac{n}{n-(i-1)}$$

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Then,

$$X = \sum^{n} X_i$$

The probability distribution of the number of tails one must flip before the first head using a weighted coin.

Let $\mathcal{E}_i$ be … in the next purchase.
Then,

$$\mathbf{Pr}\left(\mathcal{E}_i\right) = \frac{\quad - 1)}{n}$$

X_i follows a **geometric distribution**.

$$\mathbb{E}_{\mathrm{ADV}}\left[X_i\right] = \frac{1}{\mathbf{Pr}\left(\mathcal{E}_i\right)} = \frac{n}{n-(i-1)}$$

Proof.
Let X be the discrete random variable that represents the number of purchases until each of the n coupons is collected.
Le X_i be the discrete random variable that represents the number of purchases, after the $(i-1)$-th distinct coupon is collected, to collect the i-th distinct coupon.
Then,

$$X = \sum_{i=1}^{n} X_i$$

Let $\mathcal{E}_i$ be the event that one of those coupons is collected in the next purchase.
Then,

$$\mathbf{Pr}\left(\mathcal{E}_i\right) = \frac{n-(i-1)}{n}$$

X_i follows a **geometric distribution**.

$$\mathbb{E}_{\text{ADV}}\left[X_i\right] = \frac{1}{\mathbf{Pr}\left(\mathcal{E}_i\right)} = \frac{n}{n-(i-1)}$$

Then,

$$\mathbb{E}_{\text{ADV}}\left[X\right] = \mathbb{E}_{\text{ADV}}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}_{\text{ADV}}\left[X_i\right] = \sum_{i=1}^{n} \frac{n}{n-(i-1)} = n\sum_{i=1}^{n}\frac{1}{i} = nH_n$$

A lowerbound for paging

Theorem

No randomized online algorithm for paging is better than H_k-competitive in expectation, even when $M = k + 1$.

Assumptions

- Without loss of generality, $\textsc{Opt}$ and $\textsc{Alg}$ start with cache $1, 2, \ldots, k$.
- The first page requested by the adversary is page $k + 1$, causing a page fault for both $\textsc{Opt}$ and $\textsc{Alg}$.

Objective:

- Show that every deterministic online algorithm has a large expected cost compared to an optimal solution. **Yao's Principle!**

A lowerbound for paging

Theorem

No randomized online algorithm for paging is better than H_k-competitive in expectation, even when $M = k + 1$.

Input distribution

- A phase is defined as before.
- $\mathcal{I}_N$ is the set of all instances that contain N phases.
- The adversary never requests the same page twice in a row.
- The adversary requests each (other) page with probability $1/k$.

Expected cost for optimal algorithm.

- Since $M = k + 1$, LFD ensures a single page fault per phase.
- $\mathbb{E}_{\text{Adv}}\left[\text{Opt}(\mathcal{I}_n)\right] = \sum_{j=1}^{N} \mathbb{E}_{\text{Adv}}\left[\text{Opt}(P_j)\right] = \sum_{j=1}^{N} 1 = N.$

A lowerbound for paging

Theorem

No randomized online algorithm for paging is better than H_k-competitive in expectation, even when $M = k + 1$.

Expected page fault.

- Every page that is requested in some time step (not time step 1) causes a page fault with probability $\frac{1}{k}$.
- Let $|P_j|$ be the expected size of phase P_j.
- $\mathbb{E}_{\text{ADV}}\left[\text{ALG}(P_j)\right] \geq |P_j| \cdot \frac{1}{k}$.

Length of a phase.

- Phase P_j ends right before the $(k+1)$-th distinct page, since the beginning of phase P_j, will be requested.
- How long does it take in expectation until all pages $1, 2, \ldots, k, k+1$ are requested?

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Expected page fault.

- Every page that is requested in some time step (not time step 1) causes a page fault with probability $\frac{1}{k}$.
- Let $|P_j|$ be the expected size of phase P_j.
- $\mathbb{E}_{\text{ADV}}\left[\text{ALG}(P_j)\right] \geq |P_j| \cdot \frac{1}{k} = (k \cdot H_k) \cdot \frac{1}{k} = H_k$.

Length of a phase.

- Phase P_j ends right before the $(k+1)$-th distinct page, since the beginning of phase P_j, will be requested.
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A lowerbound for paging

Theorem

No randomized online algorithm for paging is better than H_k-competitive in expectation, even when $M = k + 1$.

Expected cost for every deterministic algorithm.

- $\mathbb{E}_{\text{ADV}}\left[\text{ALG}(\mathcal{I}_n)\right] = \sum_{j=1}^{N} \mathbb{E}_{\text{ADV}}\left[\text{ALG}(P_j)\right] \geq \sum_{j=1}^{N} H_k = NH_k.$

Yao's Principle.

- For every random algorithm RAND, there exists an instance $I_n \in \mathcal{I}_n$ such that
$$\frac{\mathbb{E}_{\text{RAND}}\left[\text{RAND}(I_n)\right]}{\text{OPT}(I_n)} \geq \frac{\mathbb{E}_{\text{ADV}}\left[\text{ALG}(\mathcal{I}_n)\right]}{\mathbb{E}_{\text{ADV}}\left[\text{OPT}(\mathcal{I}_n)\right]} = \frac{NH_k}{N} = H_k$$

Optimal paging

- RMark algorithm
  - Competitive ratio greater then H_k (see exercises).
  - At most $2H_k$-competitive.

- Any randomized algorithm is at least H_k-competitive.

- Does a randomized algorithm exist that is H_k-competitive in expectation?

Optimal paging

- RMark algorithm
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Optimal paging

- RMark algorithm
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- Any randomized algorithm is at least H_k-competitive.

- Does a randomized algorithm exist that is H_k-competitive in expectation?
  - Yes, the Partition algorithm (McGeoch, Lyle, Sleator, Daniel 1991).

Table of Contents

Problem definition

- Suppose you want to go skiing for as long as possible, but you do not own any skis. You have two choices.
  - Renting skis for 1 a day.
  - Buying skis for B.
- The only thing that would prevent you from skiing is the weather.
  -  You go skiing.
  -  You don't go skiing.
- Objective: Minimize the amount of money you have to spend on skis.

Ski rental, so far...

- If there are less than B sunny days, $\textsc{Opt}$ never buys.
- If there are more than B sunny days, $\textsc{Opt}$ buys on day 1.
- There exists a $(2 - \frac{1}{B})$-competitive algorithm.
- No deterministic algorithm can be better than $(2 - \frac{1}{B})$-competitive.

Ski rental, so far...

- If there are less than B sunny days, $\textsc{Opt}$ never buys.
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  - A good deterministic algorithm never buys later than day B.
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Ski rental, so far...

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- If there are more than B sunny days, $\textsc{Opt}$ buys on day 1.
  - A good deterministic algorithm never buys later than day B.
- There exists a $(2 - \frac{1}{B})$-competitive algorithm.
  - Strategy: buy on day B.
- No deterministic algorithm can be better than $(2 - \frac{1}{B})$-competitive.

A first attempt

How was the deterministic algorithm punished?

- $\textsc{Alg}$ buys on day B.
- Then there are only B sunny days.

A first attempt

How was the deterministic algorithm punished?

- ALG buys on day B.
- Then there are only B sunny days.

How can we use randomization?

- RAND buys either on day B or earlier, say $\frac{B}{2}$.
- Then the adversary should choose either B or $\frac{B}{2}$ sunny days.

A first attempt

How was the deterministic algorithm punished?

- $\textsc{Alg}$ buys on day B.
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How can we use randomization?

- $\textsc{Rand}$ buys either on day B or earlier, say $\frac{B}{2}$.
- Then the adversary should choose either B or $\frac{B}{2}$ sunny days.

Is this an improvement?

- If there are B sunny days, buying on day $\frac{B}{2}$ (compared to buying on day B) saves spending an extra $\frac{B}{2}$.
- If there are $\frac{B}{2}$ sunny days, waiting to buy on day B (compared to buying on day $\frac{B}{2}$) saves spending an extra B.

A first attempt

Procedure GAMBLE

With probability $\frac{1}{2}$ we buy on day $\frac{B}{2}$, and also with probability $\frac{1}{2}$ we buy on day B.

A first attempt

Procedure GAMBLE

With probability $\frac{1}{2}$ we buy on day $\frac{B}{2}$, and also with probability $\frac{1}{2}$ we buy on day B.

Let d be the number of sunny days.

A first attempt

Procedure GAMBLE

With probability $\frac{1}{2}$ we buy on day $\frac{B}{2}$, and also with probability $\frac{1}{2}$ we buy on day B.

Let d be the number of sunny days.
Case 1: $d \geq B$.

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- With probability $\frac{1}{2}$ $\textsc{Rand}$ rents $B - 1$ days, and then buys.

A first attempt

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$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right] = \frac{1}{2}(\frac{B}{2} - 1 + B) + \frac{1}{2}(B - 1 + B)$$

A first attempt

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$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right] = \frac{1}{2}(\frac{B}{2} - 1 + B) + \frac{1}{2}(B - 1 + B) = \frac{7B}{4} - 1$$

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- $\textsc{Opt}$ buys on day 1.

$$\textsc{Opt}(d) = B$$

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} = \frac{\frac{7B}{4} - 1}{B} = \frac{7}{4} - \frac{1}{B}$$

A first attempt

Procedure GAMBLE

With probability $\frac{1}{2}$ we buy on day $\frac{B}{2}$, and also with probability $\frac{1}{2}$ we buy on day B.

Let d be the number of sunny days.
Case 2: $\frac{B}{2} \leq d < B$.

- With probability $\frac{1}{2}$ $\textsc{Rand}$ rents $\frac{B}{2} - 1$ days, and then buys.
- With probability $\frac{1}{2}$ $\textsc{Rand}$ rents d days.

A first attempt

Procedure GAMBLE

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$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right] = \frac{1}{2}(\frac{B}{2} - 1 + B) + \frac{1}{2}d$$

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$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right] = \frac{1}{2}(\frac{B}{2} - 1 + B) + \frac{1}{2}d = \frac{3B + 2d - 2}{4}$$

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- $\textsc{Opt}$ rents for d days.

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Let d be the number of sunny days.
Case 2: $\frac{B}{2} \leq d < B$.

- With probability $\frac{1}{2}$ RAND rents $\frac{B}{2} - 1$ days, and then buys.
- With probability $\frac{1}{2}$ RAND rents d days.

$$\mathbb{E}_{\text{RAND}}\left[\text{RAND}(d)\right] = \frac{1}{2}(\frac{B}{2} - 1 + B) + \frac{1}{2}d = \frac{3B + 2d - 2}{4}$$

- OPT rents for d days.

$$\text{OPT}(d) = d$$

$$\frac{\mathbb{E}_{\text{RAND}}\left[\text{RAND}(d)\right]}{\text{OPT}(d)} = \frac{\frac{3B+2d-2}{4}}{d} = \frac{3B + 2d - 2}{4d}$$

A first attempt

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With probability $\frac{1}{2}$ we buy on day $\frac{B}{2}$, and also with probability $\frac{1}{2}$ we buy on day B.

Let d be the number of sunny days.
Case 2: $\frac{B}{2} \leq d < B$.

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The competitive ratio is maximized for $d = \frac{B}{2}$

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$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} = \frac{\frac{3B+2d-2}{4}}{d} = \frac{3B + 2d - 2}{4d} \leq \frac{4B - 2}{2B} = 2 - \frac{1}{B}$$

The competitive ratio is maximized for $d = \frac{B}{2}$

A first attempt

Procedure GAMBLE

With probability $\frac{1}{2}$ we buy on day $\frac{B}{2}$, and also with probability $\frac{1}{2}$ we buy on day B.

Let d be the number of sunny days.
Case 3: $d < \frac{B}{2}$.

- With probability 1 $\textsc{Rand}$ rents d days.

A first attempt

Procedure GAMBLE

With probability $\frac{1}{2}$ we buy on day $\frac{B}{2}$, and also with probability $\frac{1}{2}$ we buy on day B.

Let d be the number of sunny days.

Case 3: $d < \frac{B}{2}$.

- With probability 1 $\textsc{Rand}$ rents d days.

$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right] = d$$

A first attempt

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- With probability 1 $\textsc{Rand}$ rents d days.

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- $\textsc{Opt}$ rents for d days.

$$\textsc{Opt}(d) = d$$

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} = 1$$

A first attempt

Procedure GAMBLE

With probability $\frac{1}{2}$ we buy on day $\frac{B}{2}$, and also with probability $\frac{1}{2}$ we buy on day B.

Case 1: $d \geq B$

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} = \frac{7}{4} - \frac{1}{B}$$

Case 2: $\frac{B}{2} \leq d < B$

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} \leq 2 - \frac{1}{B}$$

Case 3: $d < \frac{B}{2}$

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} = 1$$

A first attempt

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With probability $\frac{1}{2}$ we buy on day $\frac{B}{2}$, and also with probability $\frac{1}{2}$ we buy on day B.

Case 1: $d \geq B$ $\qquad \dfrac{\mathbb{E}_{\text{RAND}}\left[\text{RAND}(d)\right]}{\text{OPT}(d)} = \dfrac{7}{4} - \dfrac{1}{B}$

Case 2: $\dfrac{B}{2} \leq d < B$ $\qquad \dfrac{\mathbb{E}_{\text{RAND}}\left[\text{RAND}(d)\right]}{\text{OPT}(d)} \leq 2 - \dfrac{1}{B}$

Case 3: $d < \dfrac{B}{2}$ $\qquad \dfrac{\mathbb{E}_{\text{RAND}}\left[\text{RAND}(d)\right]}{\text{OPT}(d)} = 1$

For every instance I,

$$\frac{\mathbb{E}_{\text{RAND}}\left[\text{RAND}(I)\right]}{\text{OPT}(I)} \leq 2 - \frac{1}{B}$$

A first attempt

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$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} = 1$$

For every instance I,

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(I)\right]}{\textsc{Opt}(I)} \leq 2 - \frac{1}{B}$$

No improvement over deterministic algorithm!

A first attempt

Why did this not work?

- We should not buy too early or early too often.

A first attempt

Why did this not work?

- We should not buy too early or early too often.

How can we improve?

- Buy less early.
- Buy early less.

A second attempt

Procedure EDUCATED GUESS

With probability $\frac{1}{4}$ we buy on day $\frac{B}{2}$, and with probability $\frac{3}{4}$ we buy on day B.

A second attempt

Procedure EDUCATED GUESS

With probability $\frac{1}{4}$ we buy on day $\frac{B}{2}$, and with probability $\frac{3}{4}$ we buy on day B.

Let d be the number of sunny days.

A second attempt

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With probability $\frac{1}{4}$ we buy on day $\frac{B}{2}$, and with probability $\frac{3}{4}$ we buy on day B.

Let d be the number of sunny days.
Case 1: $d \geq B$.

- With probability $\frac{1}{4}$ $\textsc{Rand}$ rents $\frac{B}{2} - 1$ days, and then buys.
- With probability $\frac{3}{4}$ $\textsc{Rand}$ rents $B - 1$ days, and then buys.

A second attempt

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$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right] = \frac{1}{4}(\frac{B}{2} - 1 + B) + \frac{3}{4}(B - 1 + B)$$

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$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right] = \frac{1}{4}\left(\frac{B}{2} - 1 + B\right) + \frac{3}{4}(B - 1 + B) = \frac{15B}{8} - 1$$

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- $\textsc{Opt}$ buys on day 1.

$$\textsc{Opt}(d) = B$$

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} = \frac{\frac{15B}{8} - 1}{B} = \frac{15}{8} - \frac{1}{B}$$

A second attempt

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With probability $\frac{1}{4}$ we buy on day $\frac{B}{2}$, and with probability $\frac{3}{4}$ we buy on day B.

Let d be the number of sunny days.
Case 2: $\frac{B}{2} \leq d < B$.

- With probability $\frac{1}{4}$ $\textsc{Rand}$ rents $\frac{B}{2} - 1$ days, and then buys.
- With probability $\frac{3}{4}$ $\textsc{Rand}$ rents d days.

A second attempt

Procedure EDUCATED GUESS

With probability $\frac{1}{4}$ we buy on day $\frac{B}{2}$, and with probability $\frac{3}{4}$ we buy on day B.

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$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right] = \frac{1}{4}(\frac{B}{2} - 1 + B) + \frac{3}{4}d$$

A second attempt

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$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right] = \frac{1}{4}(\frac{B}{2} - 1 + B) + \frac{3}{4}d = \frac{3B + 6d - 2}{8}$$

A second attempt

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- $\textsc{Opt}$ rents for d days.

$$\textsc{Opt}(d) = d$$

A second attempt

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The competitive ratio is maximized for $d = \frac{B}{2}$

A second attempt

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$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} = \frac{\frac{3B+6d-2}{8}}{d} = \frac{3B + 6d - 2}{8d} \leq \frac{6B - 2}{4B} = \frac{3}{2} - \frac{1}{2B}$$

The competitive ratio is maximized for $d = \frac{B}{2}$

A second attempt

Procedure EDUCATED GUESS

With probability $\frac{1}{4}$ we buy on day $\frac{B}{2}$, and with probability $\frac{3}{4}$ we buy on day B.

Let d be the number of sunny days.

Case 3: $d < \frac{B}{2}$.

- With probability 1 $\textsc{Rand}$ rents d days.

A second attempt

Procedure EDUCATED GUESS

With probability $\frac{1}{4}$ we buy on day $\frac{B}{2}$, and with probability $\frac{3}{4}$ we buy on day B.

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A second attempt

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$$\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right] = d$$

- $\textsc{Opt}$ rents for d days.

$$\textsc{Opt}(d) = d$$

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} = 1$$

A second attempt

Procedure EDUCATED GUESS

With probability $\frac{1}{4}$ we buy on day $\frac{B}{2}$, and with probability $\frac{3}{4}$ we buy on day B.

Case 1: $d \geq B$
$$\frac{\mathbb{E}_{\text{Rand}}\left[\text{Rand}(d)\right]}{\text{Opt}(d)} = \frac{15}{8} - \frac{1}{B}$$

Case 2: $\frac{B}{2} \leq d < B$
$$\frac{\mathbb{E}_{\text{Rand}}\left[\text{Rand}(d)\right]}{\text{Opt}(d)} \leq \frac{3}{2} - \frac{1}{2B}$$

Case 3: $d < \frac{B}{2}$
$$\frac{\mathbb{E}_{\text{Rand}}\left[\text{Rand}(d)\right]}{\text{Opt}(d)} = 1$$

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$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} = 1$$

For every instance I,
$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(I)\right]}{\textsc{Opt}(I)} \leq \frac{15}{8} - \frac{1}{B}$$

A second attempt

Procedure EDUCATED GUESS

With probability $\frac{1}{4}$ we buy on day $\frac{B}{2}$, and with probability $\frac{3}{4}$ we buy on day B.

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Case 2: $\frac{B}{2} \leq d < B$

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} \leq \frac{3}{2} - \frac{1}{2B}$$

Case 3: $d < \frac{B}{2}$

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For every instance I,

$$\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(I)\right]}{\textsc{Opt}(I)} \leq \frac{15}{8} - \frac{1}{B}$$

Slightly better than the deterministic $2 - \frac{1}{B}$ algorithm!

A lower bound for ski rental

Can we do better?

- We need to see if there exists a matching lower bound.

A lower bound for ski rental

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Yao's Principle.

- What distribution over the input should we use?

A lower bound for ski rental

Can we do better?

- We need to see if there exists a matching lower bound.

Yao's Principle.

- What distribution over the input should we use?
- Find a distribution over the inputs such that all deterministic algorithms are equally bad.

A lower bound for ski rental

Cost of the optimal algorithm.
For any input d, the cost of the optimal solution is easy to compute.

$$\textsc{Opt}(d) = \begin{cases} d & \text{if } d < B \\ B & \text{otherwise} \end{cases}$$

A lower bound for ski rental

Cost of the optimal algorithm.
For any input d, the cost of the optimal solution is easy to compute.

$$\textsc{Opt}(d) = \begin{cases} d & \text{if } d < B \\ B & \text{otherwise} \end{cases}$$

Let $\mathcal{D}$ be a probability distribution over the input. Then,

$$\mathbb{E}_{\textsc{Adv}}\left[\textsc{Opt}(\mathcal{D})\right] = \sum_{d=1}^{B-1} d \cdot \mathbf{Pr}_{\textsc{Adv}}\left(\mathcal{D} = d\right) + B \cdot \mathbf{Pr}_{\textsc{Adv}}\left(\mathcal{D} \geq B\right)$$

A lower bound for ski rental

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A lower bound for ski rental

Cost of deterministic algorithms.
Also the cost of any deterministic algorithm is easy to compute.

$$\textsc{Alg}_i(d) = \begin{cases} d & \text{if } d < i \\ i - 1 + B & \text{otherwise} \end{cases}$$

A lower bound for ski rental

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Then,

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A lower bound for ski rental

Cost of deterministic algorithms.
Also the cost of any deterministic algorithm is easy to compute.

$$\textsc{Alg}_i(d) = \begin{cases} d & \text{if } d < i \\ i - 1 + B & \text{otherwise} \end{cases}$$

Then,

$$\begin{aligned} \mathbb{E}_{\textsc{Adv}}[\textsc{Alg}_i(\mathcal{D})] &= \sum_{d=1}^{i-1} d \cdot \mathbf{Pr}_{\textsc{Adv}}(\mathcal{D} = d) + (i - 1 + B) \cdot \mathbf{Pr}_{\textsc{Adv}}(\mathcal{D} > i - 1) \\ &= \sum_{d=1}^{i-1} \mathbf{Pr}_{\textsc{Adv}}(\mathcal{D} \geq d) + B \cdot \mathbf{Pr}_{\textsc{Adv}}(\mathcal{D} > i - 1) \end{aligned}$$

A lower bound for ski rental

Given that,

$$\mathbb{E}_{\text{ADV}}\left[\text{ALG}_i(\mathcal{D})\right] = \sum_{d=1}^{i-1} \mathbf{Pr}_{\text{ADV}}\left(\mathcal{D} \geq d\right) + B \cdot \mathbf{Pr}_{\text{ADV}}\left(\mathcal{D} > i-1\right)$$

In order to find a distribution such that for any i, ALG_i performs badly, we will find the distribution which makes all algorithms perform the same in expectation.

$$\mathbb{E}_{\text{ADV}}\left[\text{ALG}_{i-1}(\mathcal{D})\right] = \mathbb{E}_{\text{ADV}}\left[\text{ALG}_i(\mathcal{D})\right]$$

A lower bound for ski rental

Given that,

$$\mathbb{E}_{\mathrm{ADV}}\left[\mathrm{ALG}_i(\mathcal{D})\right] = \sum_{d=1}^{i-1} \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq d\right) + B \cdot \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} > i-1\right)$$

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Which means that,

$$\begin{aligned}
&\sum_{d=1}^{i-2} \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq d\right) + B \cdot \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} > i-2\right) \\
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\end{aligned}$$

A lower bound for ski rental

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\end{aligned}$$

A lower bound for ski rental

Given that,

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We need,

$$B \cdot \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} > i-2\right) = \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq i-1\right) + B \cdot \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} > i-1\right)$$

A lower bound for ski rental

Given that,

$$\sum_{d=1}^{i-2} \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq d\right) + B \cdot \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} > i-2\right)$$
$$= \sum_{d=1}^{i-2} \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq d\right) + \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq i-1\right) + B \cdot \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} > i-1\right)$$

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$$\iff B \cdot \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq i-1\right) = \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq i-1\right) + B \cdot \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq i\right)$$

A lower bound for ski rental

Given that,

$$\begin{aligned}&\sum_{d=1}^{i-2} \mathbf{Pr}_{\text{ADV}}\left(\mathcal{D} \geq d\right) + B \cdot \mathbf{Pr}_{\text{ADV}}\left(\mathcal{D} > i-2\right) \\ =&\sum_{d=1}^{i-2} \mathbf{Pr}_{\text{ADV}}\left(\mathcal{D} \geq d\right) + \mathbf{Pr}_{\text{ADV}}\left(\mathcal{D} \geq i-1\right) + B \cdot \mathbf{Pr}_{\text{ADV}}\left(\mathcal{D} > i-1\right)\end{aligned}$$

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A lower bound for ski rental

Given that,

$$\begin{aligned}&\sum_{d=1}^{i-2} \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq d\right) + B \cdot \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} > i-2\right) \\ =&\sum_{d=1}^{i-2} \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq d\right) + \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq i-1\right) + B \cdot \mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} > i-1\right)\end{aligned}$$

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This is fulfilled when we set,

$$\mathbf{Pr}_{\mathrm{ADV}}\left(\mathcal{D} \geq d\right) = \left(1 - \frac{1}{B}\right)^{d-1} \qquad \text{for all } d \geq 1.$$

A lower bound for ski rental

Given that,

$$\mathbf{Pr}_{\textsc{Adv}}\left(\mathcal{D} \geq d\right) = \left(1 - \frac{1}{B}\right)^{d-1} \qquad \text{for all } d \geq 1.$$

The cost for $\textsc{Opt}$ equals

$$\mathbb{E}_{\textsc{Adv}}\left[\textsc{Opt}(\mathcal{D})\right] = \sum_{d=1}^{B} \mathbf{Pr}_{\textsc{Adv}}\left(\mathcal{D} \geq d\right)$$

A lower bound for ski rental

Given that,

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$$\begin{aligned}
\mathbb{E}_{\text{ADV}}\left[\textsc{Opt}(\mathcal{D})\right] &= \sum_{d=1}^{B} \mathbf{Pr}_{\text{ADV}}\left(\mathcal{D} \geq d\right) \\
&= \sum_{d=1}^{B} \left(1 - \frac{1}{B}\right)^{d-1} \\
&= B \cdot \left(1 - \left(1 - \frac{1}{B}\right)^{B}\right)
\end{aligned}$$

A lower bound for ski rental

Given that,

$$\mathbf{Pr}_{\text{ADV}}(\mathcal{D} \geq d) = \left(1 - \frac{1}{B}\right)^{d-1} \qquad \text{for all } d \geq 1.$$

The cost for $\textsc{Alg}_i$ equals

$$\mathbb{E}_{\text{ADV}}[\textsc{Alg}_i(\mathcal{D})] = \sum_{d=1}^{i-1} \mathbf{Pr}_{\text{ADV}}(\mathcal{D} \geq d) + B \cdot \mathbf{Pr}_{\text{ADV}}(\mathcal{D} > i - 1)$$

A lower bound for ski rental

Given that,

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A lower bound for ski rental

Given that,

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The cost for $\textsc{Alg}_i$ equals

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&= \sum_{d=1}^{i-1} \left(1 - \frac{1}{B}\right)^{d-1} + B \cdot \left(1 - \frac{1}{B}\right)^{i-1} \\
&= B \cdot \left(1 - \left(1 - \frac{1}{B}\right)^{i-1}\right) + B \cdot \left(1 - \frac{1}{B}\right)^{i-1} \\
&= B
\end{aligned}$$

A lower bound for ski rental

Using Yao's Principle.

- For every random algorithm $\textsc{Rand}$, there exists an instance $d \in \mathcal{D}$ such that

$$
\begin{aligned}
&\frac{\mathbb{E}_{\textsc{Rand}}\left[\textsc{Rand}(d)\right]}{\textsc{Opt}(d)} \\
\geq &\frac{\min_i \mathbb{E}_{\textsc{Adv}}\left[\textsc{Alg}_i(\mathcal{D})\right]}{\mathbb{E}_{\textsc{Adv}}\left[\textsc{Opt}(\mathcal{D})\right]} \\
= &\frac{B}{B \cdot \left(1 - \left(1 - \frac{1}{B}\right)^B\right)} \\
= &\left(1 - \left(1 - \frac{1}{B}\right)^B\right)^{-1}
\end{aligned}
$$

A lower bound for ski rental

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= &\left(1 - \left(1 - \frac{1}{B}\right)^B\right)^{-1}
\end{aligned}
$$

For large B,

$$
\lim_{B \to \infty} \left(1 - \left(1 - \frac{1}{B}\right)^B\right)^{-1} = \frac{e}{e-1} \approx 1.582
$$

A lower bound for ski rental

What did we do?

- Computed an expression for the expected costs of $\textsc{Alg}_i$ and $\textsc{Opt}$.
- Found the distribution that made the expected costs of all $\textsc{Alg}_i$ equal.
- Computed the expected costs following this distribution for $\textsc{Alg}_i$ and $\textsc{Opt}$.
- Applied Yao's Principle.

A lower bound for ski rental

What did we do?

- Computed an expression for the expected costs of $\textsc{Alg}_i$ and $\textsc{Opt}$.
- Found the distribution that made the expected costs of all $\textsc{Alg}_i$ equal.
  - If you know a (good) distribution, finding a lower bound using Yao's Principle is not that hard. **Try it out in the exercises!**
- Computed the expected costs following this distribution for $\textsc{Alg}_i$ and $\textsc{Opt}$.
- Applied Yao's Principle.

The optimal randomized algorithm

- Our expected $(\frac{15}{8} - \frac{1}{B})$-competitive approach was not optimal.
- Optimal solution is $(\frac{e}{e-1})$-competitive in expectation.
- It selects a day in $\{1, 2, \ldots, B\}$ with increasing probability towards B.

Overview

- For paging the RMARK algorithm is close to being optimal.
- For ski rental, even with randomization, buying early is risky.
- **Yao's Principle is a strong tool to compute lower bounds for randomized algorithms.**