Exercise: Randomized Online Algorithms

1 Better randomized ski rental algorithms

Consider the ski rental problem. Suppose a random algorithm would buy either on day $\frac{3B}{4}$ or on day B with probability $\frac{1}{2}$ each. What is the expected competitive ratio of this algorithm?

Let d be the number of sunny days.

Case 1: $d \geq B$

$$\mathbb{E}_{\textsc{Rand}}[\textsc{Rand}(d)] = \frac{1}{2} \cdot (\frac{3B}{4} - 1 + B) + \frac{1}{2} \cdot (B - 1 + B) = \frac{15B}{8} - 1$$

$$\textsc{Opt}(d) = B$$

$$\frac{\mathbb{E}_{\textsc{Rand}}[\textsc{Rand}(d)]}{\textsc{Opt}(d)} = \frac{\frac{15B}{8} - 1}{B} = \frac{15}{8} - \frac{1}{B}$$

Case 2: $\frac{3B}{4} \leq d < B$

$$\mathbb{E}_{\textsc{Rand}}[\textsc{Rand}(d)] = \frac{1}{2} \cdot (\frac{3B}{4} - 1 + B) + \frac{1}{2} \cdot d = \frac{7B + 4d - 4}{8}$$

$$\textsc{Opt}(d) = d$$

$$\frac{\mathbb{E}_{\textsc{Rand}}[\textsc{Rand}(d)]}{\textsc{Opt}(d)} = \frac{\frac{7B+4d-4}{8}}{d} \leq \frac{\frac{10B-4}{8}}{\frac{3B}{4}} = \frac{5}{3} - \frac{2}{3B}$$

Case 3: $d < \frac{3B}{4}$

$$\mathbb{E}_{\textsc{Rand}}[\textsc{Rand}(d)] = d$$

$$\textsc{Opt}(d) = d$$

$$\frac{\mathbb{E}_{\textsc{Rand}}[\textsc{Rand}(d)]}{\textsc{Opt}(d)} = 1$$

Hence we conclude that for any d, the expected competitive ratio is

$$\frac{\mathbb{E}_{\textsc{Rand}}[\textsc{Rand}(d)]}{\textsc{Opt}(d)} \leq \frac{15}{8} - \frac{1}{B}$$

2 A lower bound for ski the ski rental problem.

Consider the ski rental problem. Given that the the number of days d, with good weather is either $\frac{B}{2}$ or $\frac{3B}{2}$ with probability $\frac{1}{2}$, that is,

$$\mathbf{Pr}(d = \frac{B}{2}) = \mathbf{Pr}(d = \frac{3B}{2}) = \frac{1}{2}$$

We will prove, using Yao's Principle, that the competitive ratio of any randomized algorithm is at least $\frac{4}{3}$.

(a) Compute the expected cost of OPT.
$\mathbb{E}_{\text{ADV}}[\text{OPT}(\mathcal{D})] = \frac{1}{2} \cdot \frac{B}{2} + \frac{1}{2} \cdot B = \frac{3B}{4}$.

(b) Compute the expected cost of ALG_i for $i \leq \frac{B}{2}$.
$\mathbb{E}_{\text{ADV}}[\text{ALG}_i(\mathcal{D})] = \frac{1}{2} \cdot (i - 1 + B) + \frac{1}{2} \cdot (i - 1 + B) = (i - 1 + B)$.

(c) Compute the expected cost of ALG_i for $\frac{B}{2} < i \leq \frac{3B}{2}$.
$\mathbb{E}_{\text{ADV}}[\text{ALG}_i(\mathcal{D})] = \frac{1}{2} \cdot \frac{B}{2} + \frac{1}{2} \cdot (i - 1 + B) = \frac{3B}{4} + \frac{i-1}{2}$.

(d) Compute the expected cost of ALG_i for $i > \frac{3B}{2}$.
$\mathbb{E}_{\text{ADV}}[\text{ALG}_i(\mathcal{D})] = \frac{1}{2} \cdot \frac{B}{2} + \frac{1}{2} \cdot \frac{3B}{2} = B$.

(e) Prove, using Yao's Principle, that he competitive ratio of any randomized algorithm is at least $\frac{4}{3}$.
The algorithms that minimize the expected cost are $\text{ALG}_{\frac{B}{2}+1}$ and all ALG_i for $i > \frac{3B}{4}$. Then,

$$\min_i \mathbb{E}_{\text{ADV}}[\text{ALG}_i(\mathcal{D})] = B$$

And,

$$\mathbb{E}_{\text{ADV}}[\text{OPT}(\mathcal{D})] = \frac{3B}{4}$$

Then, it follows from Yao's Principle that there exists an input I, such that,

$$\frac{\mathbb{E}_{\text{RAND}}[\text{RAND}(I)]}{\text{OPT}(I)} \geq \frac{\min_i \mathbb{E}_{\text{ADV}}[\text{ALG}_i(\mathcal{D})]}{\mathbb{E}_{\text{ADV}}[\text{OPT}(\mathcal{D})]} \geq \frac{B}{\frac{3B}{4}} = \frac{4}{3}$$

3 Online Vertex Cover.

Let $G = (V, E)$ be an unweighted undirected graph. Consider the following online version of the *minimum vertex cover* problem. Initially we are given the set of vertices V and an empty vertex cover $S = \emptyset$. Then, the edges appear one-by-one in an online fashion. When a new edge (u, v) appears, the algorithm needs to guarantee that the edge is covered (i.e., if this is not already the case, at least one of the two nodes u and v needs to be added to S). Once a node is in S it cannot be removed from S.

Consider the following distribution, with probability $\frac{1}{2}$ the adversary picks $I_1 = \{(u, v), (u, w)\}$ and with probability $\frac{1}{2}$ the adversary picks $I_2 = \{(u, v), (v, w)\}$.

(a) Compute the expected cost of OPT.
$\mathbb{E}_{\text{ADV}}[\text{OPT}(\mathcal{I})] = \mathbf{Pr}_{\text{ADV}}(I_1) \cdot \mathbb{E}_{\text{ADV}}[\text{OPT}(I_1)] + \mathbf{Pr}_{\text{ADV}}(I_2) \cdot \mathbb{E}_{\text{ADV}}[\text{OPT}(I_2)] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1.$

Let ALG_u be the algorithm that when a new edge (u, v) is not already covered adds vertex u to the vertex cover.

(b) Compute the expected cost of ALG_u.
$\mathbb{E}_{\text{ADV}}[\text{ALG}_u(\mathcal{I})] = \mathbf{Pr}_{\text{ADV}}(I_1) \cdot \mathbb{E}_{\text{ADV}}[\text{ALG}_u(I_1)] + \mathbf{Pr}_{\text{ADV}}(I_2) \cdot \mathbb{E}_{\text{ADV}}[\text{ALG}_u(I_2)] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = \frac{3}{2}.$

Let ALG_v be the algorithm that when a new edge (u, v) is not already covered adds vertex v to the vertex cover.

(c) Compute the expected cost of ALG_v.
$\mathbb{E}_{\text{ADV}}[\text{ALG}_v(\mathcal{I})] = \mathbf{Pr}_{\text{ADV}}(I_1) \cdot \mathbb{E}_{\text{ADV}}[\text{ALG}_v(I_1)] + \mathbf{Pr}_{\text{ADV}}(I_2) \cdot \mathbb{E}_{\text{ADV}}[\text{ALG}_v(I_2)] = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{3}{2}.$

Let ALG_{uv} be the algorithm that when a new edge (u, v) is not already covered adds both vertices u and v to the vertex cover.

(d) Compute the expected cost of ALG_{uv}.
$\mathbb{E}_{\text{ADV}}[\text{ALG}_{uv}(\mathcal{I})] = \mathbf{Pr}_{\text{ADV}}(I_1) \cdot \mathbb{E}_{\text{ADV}}[\text{ALG}_{uv}(I_1)] + \mathbf{Pr}_{\text{ADV}}(I_2) \cdot \mathbb{E}_{\text{ADV}}[\text{ALG}_{uv}(I_2)] = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2.$

(e) Prove, using Yao's Principle, that no randomized online algorithm for the online minimum vertex cover problem has a competitive ratio less than $\frac{3}{2}$.

Observe that since this distribution of instances contains no more than 2 edges per instance, it is irrelevant which of the two vertices is selected when the second edge arrives. Hence the cost of any algorithm is determined by which vertex is added to the vertex cover when the first edge arrives. Then, it follows that no algorithm can be better on this distribution of instances than algorithms $\textsc{Alg}_u$ or $\textsc{Alg}_v$. Then,

$$\min_i \mathbb{E}_{\textsc{Adv}}[\textsc{Alg}_i(\mathcal{I})] = \frac{3}{2}$$

And,

$$\mathbb{E}_{\textsc{Adv}}[\textsc{Opt}(\mathcal{I})] = 1$$

Then, it follows from Yao's Principle that there exists an input I, such that,

$$\frac{\mathbb{E}_{\textsc{Rand}}[\textsc{Rand}(I)]}{\textsc{Opt}(I)} \geq \frac{\min_i \mathbb{E}_{\textsc{Adv}}[\textsc{Alg}_i(\mathcal{I})]}{\mathbb{E}_{\textsc{Adv}}[\textsc{Opt}(\mathcal{I})]} \geq \frac{\frac{3}{2}}{1} = \frac{3}{2}$$