Exercise: Randomized Online Algorithms

1 Better randomized ski rental algorithms

Consider the ski rental problem. Suppose a random algorithm would buy either on day $\frac{3B}{4}$ or on day B with probability $\frac{1}{2}$ each. What is the expected competitive ratio of this algorithm?

2 A lower bound for ski the ski rental problem.

Consider the ski rental problem. Given that the the number of days d, with good weather is either $\frac{B}{2}$ or $\frac{3B}{2}$ with probability $\frac{1}{2}$, that is,

$$\mathbf{Pr}(d = \frac{B}{2}) = \mathbf{Pr}(d = \frac{3B}{2}) = \frac{1}{2}$$

We will prove, using Yao's Principle, that the competitive ratio of any randomized algorithm is at least $\frac{4}{3}$.

(a) Compute the expected cost of $\textsc{Opt}$.

(b) Compute the expected cost of $\textsc{Alg}_i$ for $i \leq \frac{B}{2}$.

(c) Compute the expected cost of $\textsc{Alg}_i$ for $\frac{B}{2} < i \leq \frac{3B}{2}$.

(d) Compute the expected cost of $\textsc{Alg}_i$ for $i > \frac{3B}{2}$.

(e) Prove, using Yao's Principle, that he competitive ratio of any randomized algorithm is at least $\frac{4}{3}$.

3 Online Vertex Cover.

Let $G = (V, E)$ be an unweighted undirected graph. Consider the following online version of the *minimum vertex cover* problem. Initially we are given the set of vertices V and an empty vertex cover $S = \emptyset$. Then, the edges appear one-by-one in an online fashion. When a new edge (u, v) appears, the algorithm needs to guarantee that the edge is covered (i.e., if this is not already the case, at least one of the two nodes u and v needs to be added to S). Once a node is in S it cannot be removed from S.

Consider the following distribution, with probability $\frac{1}{2}$ the adversary picks $I_1 = \{(u,v),(u,w)\}$ and with probability $\frac{1}{2}$ the adversary picks $I_2 = \{(u,v),(v,w)\}$.

(a) Compute the expected cost of Opt.

Let Alg_u be the algorithm that when a new edge (u,v) is not already covered adds vertex u to the vertex cover.

(b) Compute the expected cost of Alg_u.
Let Alg_v be the algorithm that when a new edge (u,v) is not already covered adds vertex v to the vertex cover.

(c) Compute the expected cost of Alg_v.

Let Alg_{uv} be the algorithm that when a new edge (u,v) is not already covered adds both vertices u and v to the vertex cover.

(d) Compute the expected cost of Alg_{uv}.

(e) Prove, using Yao's Principle, that no randomized online algorithm for the online minimum vertex cover problem has a competitive ratio less than $\frac{3}{2}$.