Exercise 1: Ever Given

1 Ever Given

In March 2021, a container ship *Ever Given* blocked the Suez Canal. Consider you are the captain of one of the ships that is waiting to pass through Suez Canal. You want to arrive at your destination as soon as possible. However, because of the obstruction, your ship is idle near the Suez Canal, and you have no idea when the canal will be available again. You have two options:

- to wait until the canal is available, and you can pass through the canal in F unit of time; or
- to go around Africa via the Cape of Good Hope, and you can arrive your destination in S units of time.

Note that F stands for fast, S stands for slow, and $F < S$.

Let p be the prediction about the time when the canal is available. Answer the following questions:

1. Consider the following algorithm. Analyze the consistency and robustness.

Algorithm 1 Algorithm 1 (p)

```
if p ≥ S − F then
    Take the long path at time 0
else                                                          ▷ p < S − F
    Wait until time S − F (and then take the long path at time S − F)
```

Let t be the actual time when the canal is available.

Consistency. If $t \geq S - F$, a correct advice $p \geq S - F$. In this case, $\frac{\mathrm{ALG}(S,F,t,p)}{\mathrm{OPT}(S,F,t)} = \frac{S}{S} = 1$. If $t < S - F$, a correct advice $p < S - F$. In this case, $\frac{\mathrm{ALG}(S,F,t,p)}{\mathrm{OPT}(S,F,t)} = \frac{t+F}{t+F} = 1$. Therefore, the algorithm is 1-consistent.

Robustness. If $t \geq S - F$, a wrong advice $p < S - F$. In this case, $\frac{\mathrm{ALG}(S,F,t,p)}{\mathrm{OPT}(S,F,t)} = \frac{(S-F)+S}{S} = 2 - \frac{F}{S}$. If $t < S - F$, a wrong advice $p \geq S - F$. In this case, $\frac{\mathrm{ALG}(S,F,t,p)}{\mathrm{OPT}(S,F,t)} = \frac{S}{t+F} \leq \frac{S}{F}$. Therefore, the algorithm is $\max\{2 - \frac{F}{S}, \frac{S}{F}\} = \frac{S}{F}$-robust.

2. Let $k \in [0, S - F]$ be a trustness parameter. Consider the following algorithm. Analyze the consistency and robustness.

Algorithm 2 Algorithm 2 (p)

```
if p ≥ S − F then
    Wait until time k (and then take the long path at time k)
else                                                          ▷ p < S − F
    Wait until time S − F (and then take the long path at time S − F)
```

Let t be the actual time when the canal is available.

Consistency. If $t \geq S-F$, a correct advice $p \geq S-F$. In this case, $\frac{\mathrm{ALG}(S,F,t,p)}{\mathrm{OPT}(S,F,t)} = \frac{k+S}{S} = 1 + \frac{k}{S}$. If $t < S - F$, a correct advice $p < S - F$. In this case, $\frac{\mathrm{ALG}(S,F,t,p)}{\mathrm{OPT}(S,F,t)} = \frac{t+F}{t+F} = 1$. Therefore, the algorithm is $\max\{1, 1 + \frac{k}{S}\} = (1 + \frac{k}{S})$-consistent.

Robustness. If $t \geq S-F$, a wrong advice $p < S-F$. In this case, $\frac{\text{ALG}(S,F,t,p)}{\text{OPT}(S,F,t)} = \frac{(S-F)+S}{S} = 2-\frac{F}{S}$. If $t < S-F$, a wrong advice $p \geq S-F$. In this case, $\frac{\text{ALG}(S,F,t,p)}{\text{OPT}(S,F,t)} = \frac{t+S}{t+F} \leq \frac{k+S}{k+F} = 1+\frac{S-F}{k+F}$. Therefore, the algorithm is $\max\{2-\frac{F}{S}, 1+\frac{S-F}{k+F}\} = (1+\frac{S-F}{k+F})$-robust.

3. Follow Question 2. Show that the algorithm is Pareto optimal.

   Let A be an algorithm that is at most $(1+\frac{k}{S})$-consistent. Given a prediction $p > S-F$ and instance $t = S$ (where t is the actual time when the Canal is available), the algorithm A's cost should be at most $(1+\frac{k}{S}) \cdot S = S+k$ since $\text{OPT}(t) = S$. Therefore, A must turn to the long path no later than k.

   Now, consider the same prediction ($p > S-F$) but with $t = t' + \epsilon$, where t' is the time when A turns to the long path. In this case, $\text{OPT} = t' + \epsilon + F$. Since A sees the same prediction, it behaves the same and turns to the long path at time $t' \leq k$. The robustness of A is then $\frac{S+t'}{t'+\epsilon+F} \approx \frac{t'+F+S-F}{t'+F} = 1+\frac{S-F}{t'+F} \geq 1+\frac{S-F}{k+F}$.

   That is, we prove that for any $(1+\frac{k}{S})$-consistent algorithm, its robustness must be at least $1+\frac{S-F}{k+F}$. Therefore, the algorithm in Question 2 is Pareto optimal.

4. Let $\alpha \in [0,1]$ be a trustness parameter. Consider the following algorithm. Show that the consistency of the algorithm is $2-\alpha$ and the robustness of the algorithm is $\frac{2-\alpha}{1-\alpha}$.

Algorithm 3 Algorithm 3 (p)

```
if p ≥ S − F then
    Wait until time (S−F)·(1−α) (and then take the long path at time (S−F)·(1−α))
else                                                                  ▷ p < S − F
    Wait until time (S−F)/(1−α) (and then take the long path at time (S−F)/(1−α))
```

   Let t be the actual time when the canal is available.

   Consistency. If $t \geq S-F$, a correct advice $p \geq S-F$. In this case, $\frac{\text{ALG}(S,F,t,p)}{\text{OPT}(S,F,t)} = \frac{(S-F)(1-\alpha)+S}{S} \leq \frac{S\cdot(1-\alpha)+S}{S} = (2-\alpha)$. If $t < S-F$, a correct advice $p < S-F$. In this case, $\frac{\text{ALG}(S,F,t,p)}{\text{OPT}(S,F,t)} = \frac{t+F}{t+F} = 1$. Therefore, the algorithm is $\max\{2-\alpha, 1\} = (2-\alpha)$-consistent.

   Robustness. If $t \geq S-F$, a wrong advice $p < S-F$. In this case, there are two subcases: $t < \frac{S-F}{1-\alpha}$, or $t \geq \frac{S-F}{1-\alpha}$. In the first case, $\frac{\text{ALG}(S,F,t,p)}{\text{OPT}(S,F,t)} = \frac{t+F}{S} < \frac{\frac{S-F}{1-\alpha}+F}{S} = \frac{S-F+F\cdot(1-\alpha)}{(1-\alpha)\cdot S} = \frac{S-\alpha\cdot F}{(1-\alpha)\cdot S} \leq \frac{1}{1-\alpha}$. In the second case, $\frac{\text{ALG}(S,F,t,p)}{\text{OPT}(S,F,t)} = \frac{\frac{S-F}{1-\alpha}+S}{S} = \frac{S-F+S\cdot(1-\alpha)}{(1-\alpha)\cdot S} \leq \frac{(1-\alpha)\cdot S+S}{(1-\alpha)\cdot S} = \frac{2-\alpha}{1-\alpha}$.

   If $t < S-F$, a wrong advice $p \geq S-F$. In this case, $\frac{\text{ALG}(S,F,t,p)}{\text{OPT}(S,F,t)} = \frac{(S-F)(1-\alpha)+S}{(S-F)(1-\alpha)+F} = \frac{(S-F)(1-\alpha)+F+(S-F)}{(S-F)(1-\alpha)+F} = 1+\frac{S-F}{(S-F)(1-\alpha)+F} \leq 1+\frac{S-F}{(S-F)(1-\alpha)} = 1+\frac{1}{1-\alpha} = \frac{2-\alpha}{1-\alpha}$. Therefore, the algorithm is $\max\{\frac{1}{1-\alpha}, \frac{2-\alpha}{1-\alpha}\} = \frac{2-\alpha}{1-\alpha}$-robust.

5. Follow Question 4. Let the error measure $\eta = |p-t|$, where p and t are the predicted and the actual time when the canal is available, respectively. Analyze Algorithm 3 and show that the consistency is $2-\alpha$ and the robustness is $\max\{1+\frac{\eta}{\alpha\cdot\text{OPT}}, 2-\alpha+\frac{(2-\alpha)\cdot\eta}{\text{OPT}}\}$, where OPT is the optimal cost.

   Consistency. It's the same with Question 3.

   Robustness. If $t \geq S-F$, a wrong advice $p < S-F$. In this case, $\text{ALG}(S,F,t,p) = \frac{S-F}{1-\alpha}+S$ and $\text{OPT}(S,F,t) = S$. Moreover, $\eta = t-p \geq \frac{S-F}{1-\alpha} - p \geq \frac{S-F}{1-\alpha} - (S-F) = (S-F)\cdot\frac{\alpha}{1-\alpha}$. Thus, $\frac{\text{ALG}(S,F,t,p)}{\text{OPT}(S,F,t)} \leq \frac{\frac{S-F}{1-\alpha}+S}{S} \leq \frac{\frac{\eta}{\alpha}+S}{S} = \frac{\eta}{\alpha\cdot\text{OPT}}+1$.

   If $t < S-F$, a wrong advice $p \geq S-F$. In this case, $\text{OPT} = t+F$, and $\eta = p-t$. Thus, $S-F \leq p = \eta+t = \text{OPT}-F+\eta$. Therefore, $\text{ALG} \leq (S-F)(1-\alpha)+S = \text{OPT}+(S-F) \leq \text{OPT}+(\text{OPT}-F+\eta) \leq 2\text{OPT}+\eta$.